Fractional Degree-two Polytopes and Ideal Polytopes of Bidirected Graphs[†]

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Abstract

An inequality in n variables $x_1, \cdots, x_n$ is of degree-two if it is either $x_i + x_j \leq 1$, $-x_i - x_j \leq -1$ or $x_i - x_j \leq 0$ for some $i, j = 1, \cdots, n$. In many combinatorial optimization problems the feasible solutions can be described as the 0-1 solutions of degree-two constraints. In this paper we consider the LP relaxation of the 0-1 solutions of degree-two inequalities, which we call a fractional degree-two polytope. We show that any fractional degree-two polytope is isomorphic to an ideal polytope of a bidirected graph. The isomorphism shows an interesting relationship between the linear programs over fractional degree-two polytopes and the minimum-weight ideal problems for bidirected graphs. Also, any fractional degree-two polytope is shown to be half-integral and we give characterizations for a fractional degree-two polytope to be integral.

1. Introduction

Edmonds and Johnson [5] first introduced the concept of bidirected graph and showed that some integer programming problems on bidirected graphs can be well-solved. A *bidirected graph* $G = (V, A; \partial)$ is a graph with a vertex set V, an arc set

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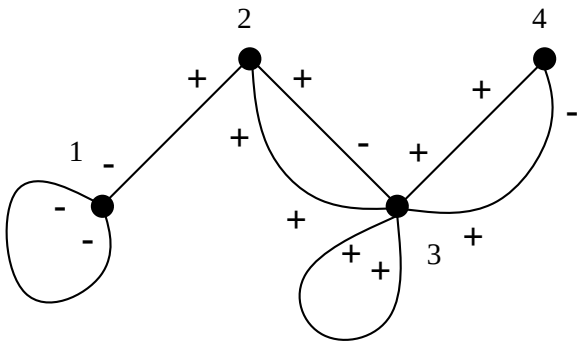

Figure 1.1: An Example of a Bidirected Graph.

A and a boundary operator $\partial : A \to \mathbf{Z}^V$, where for each arc $a \in A$ there exist $v, w \in V$ (called *end-vertices* of a) such that one of the following three holds:

(1) $\partial a = v + w$ (arc a has two tails at v and w),

(2) $\partial a = -v - w$ (arc a has two heads at v and w),

(3) $\partial a = v - w$ (arc a has a tail at v and a head at w).

Here, each $\partial a \in \mathbf{Z}^V$ is represented by an element of a free module with a base V. If $v = w$ in (1)$\sim$(3), then arc a is called a *selfloop*. For simplicity we do not allow any selfloop of type (3) in the following. See Figure 1.1 for an example of a bidirected graph with $V = \{1, 2, 3, 4\}$.

An inequality in n variables $x_1, \cdots, x_n$ is of *degree-two* if it is either $x_i + x_j \leq 1$, $-x_i - x_j \leq -1$ or $x_i - x_j \leq 0$ for some $i, j = 1, \cdots, n$. For example, the following is a system of degree-two inequalities:

$$\begin{array}{rrrrrr} -2x_1 & & & & \leq & -1, \\ -x_1 & +x_2 & & & \leq & 0, \\ & x_2 & +x_3 & & \leq & 1, \\ & x_2 & -x_3 & & \leq & 0, \\ & & x_3 & +x_4 & \leq & 1, \\ & & x_3 & -x_4 & \leq & 0, \\ & & 2x_3 & & \leq & 1. \end{array} \tag{1.1}$$

The 0-1 solutions of degree-two inequalities are of special interest. The stable sets, the node covers, the order ideals of a (directed) graph are described as the 0-1 solutions of systems of degree-two inequalities. Other examples of the 0-1 solutions of a degree-two inequalities can be found in many contexts, such as quadratic Boolean equations [7], linear and quadratic 0-1 programming [6], and logic [14]. In studies of quadratic Boolean equation (see e.g., [4] and [7]), bidirected graphs appeared implicitly as *implication graphs*, which was called signed covering graphs

by T. Zaslavsky [16]. It should be also noted that a system of degree-two constraints is, in disguise, a *complete set of implicants* with their lengths at most two (see [10], [9], [11]).

In [12] Johnson and Padberg considered the polytope defined as the convex hull of the 0-1 solutions of a system of degree-two inequalities as a generalization of a stable set polytope and gave a class of facet-inducing inequalities. In this chapter, we consider a relaxation of the 0-1 solutions of degree-two inequalities, namely, we consider the solution set of degree-two inequalities and the inequalities $0 \leq x_j \leq 1$ $(j = 1, \cdots, n)$, which we call a fractional degree-two polytope.

In Section 2, we show that there is an isomorphism between a fractional degree-two polytope and an ideal polytope. Also, the integrality of the ideal polytope of any bidirected graph is shown, and the half-integrality of any fractional degree-two polytope follows. It will be shown that the linear programs over fractional degree-two polytopes can be reduced to minimum-weight ideal problems for bidirected graphs. In Section 3, we give some necessary and sufficient condition for a fractional degree-two polytope to be integral. Finally in Section 4, we consider some set systems characterized by complete sets of implicants as an application.

2. Fractional Degree-two Polytopes and Ideal Polytopes

An *ideal polytope* $\mathrm{IP}(G)$ associated with a bidirected graph $G = (V, A; \partial)$ is defined as the solution set of the system

$$\langle \partial a, x \rangle \leq 0 \quad (a \in A), \tag{2.1}$$

$$-1 \leq x(v) \leq 1 \quad (v \in V) \tag{2.2}$$

of inequalities.

We denote by 3^V the set of all the ordered pair of disjoint subsets of V, i.e., $3^V = \{(X, Y) \mid X, Y \subseteq V, X \cap Y = \emptyset\}$. Each element (X, Y) of 3^V can be identified with the $\{0, \pm 1\}$-vector $\chi_{(X,Y)}$ defined by

$$\chi_{(X,Y)}(v) = \begin{cases} 1 & \text{if } v \in X \\ -1 & \text{if } v \in Y \\ 0 & \text{otherwise} \end{cases} \qquad (v \in V). \tag{2.3}$$

Because of this we call each element of 3^V a *signed subset* of V.

An integral solution x of (2.1)$\sim$(2.2) is made correspond to a signed subset (X, Y) of V as

$$(X, Y) = (\{v \mid v \in V, x(v) = 1\}, \{v \mid v \in V, x(v) = -1\}). \tag{2.4}$$

and is called an *ideal* of G (see [15],[2]). An ideal (X, Y) of G is called *spanning* if $X \cup Y = V$. Let us denote by $\mathcal{I}(G)$ the set of all the ideals of G.

Given a bidirected graph $G = (V, A; \partial)$ and a weight function $w : V \to \mathbf{R}$, the *minimum-weight ideal problem* ([3]) is defined as follows.

$$\begin{array}{lll} (\mathrm{P}_w): & \text{Minimize} & w(X,Y) = \sum_{v\in X} w(v) - \sum_{v\in Y} w(v) \\ & \text{subject to} & (X,Y) \in \mathcal{I}(G). \end{array} \tag{2.5}$$

Ando, Fujishige and Nemoto [3] showed that the minimum-weight ideal problems for bidirected graphs can be reduced to the minimum-weight order ideal problem for (ordinary) directed graphs, and hence, can further be reduced to the minimum cut problem.

For a system $\sum_{l=1}^{n} \gamma_{kl} x_l \leq \beta_k$ $(k = 1, \cdots, m)$ of degree-two inequalities in n variables $x_1, \cdots, x_n$ we naturally associate a bidirected graph $G = (V, A; \partial)$ as follows ([12]): $V = \{v_1, \cdots, v_n\}$, $A = \{a_1, \cdots, a_m\}$ and ∂ is defined as

$$\partial a_k = \begin{cases} v_i + v_j & \\ -v_i - v_j & \text{if} \\ v_i - v_j & \end{cases} \begin{cases} \text{the } k\text{-th inequality is } x_i + x_j \leq 1 \\ \text{the } k\text{-th inequality is } -x_i - x_j \leq -1 \\ \text{the } k\text{-th inequality is } x_i - x_j \leq 0 \end{cases} \quad (k = 1, \cdots, m). \tag{2.6}$$

For our example (1.1) the bidirected graph in Figure 1.1 corresponds. Now the system $\sum_{l=1}^{n} \gamma_{kl} x_l \leq \beta_k$ $(k = 1, \cdots, m)$ described in terms of bidirected graph G as

$$\langle \partial a, x \rangle \ \leq \ \frac{1}{2} \langle \partial a, \mathbf{1}_V \rangle \quad (a \in A), \tag{2.7}$$

where $\mathbf{1}_V \in \mathbf{R}^V$ stands for the all-1 column vector, $\langle \cdot, \cdot \rangle$ is the (canonical) inner product, and ∂a should be regarded as a vector in $\mathbf{R}^V$. Conversely, given any bidirected graph $G = (V, A; \partial)$, the system (2.7) of inequalities is of degree-two. Hence, from now on, we always associate a bidirected graph G with a system of degree-two inequalities.

Given a bidirected graph $G = (V, A; \partial)$, we call the solution set of the system

$$\langle \partial a, x \rangle \ \leq \ \tfrac{1}{2} \langle \partial a, \mathbf{1}_V \rangle \quad (a \in A), \tag{2.8}$$

$$0 \leq x(v) \leq 1 \quad (v \in V) \tag{2.9}$$

of inequalities the *fractional degree-two polytope* associated with bidirected graph $G = (V, A; \partial)$ and denote it by FD2P(G). It should be noted that if all the arcs of G are of type (1) then G can be regarded as an undirected graph and FD2P(G) is the *fractional stable set polytope* of G and the integral points of FD2P(G) are the characteristic vectors of the *stable sets* of G. Also, if all the arcs of G are of type (3), then G is an ordinary directed graph and FD2P(G) is the *order ideal polytope* of G and the integral points of FD2P(G) are the characteristic vectors of the *order ideals* of G.

It follows from the definitions that

Proposition 2.1: *For any bidirected graph $G = (V, A; \partial)$ we have $x \in \mathrm{FD2P}(G)$ if and only if $2x - \mathbf{1}_V \in \mathrm{IP}(G)$.* □

Also, we can easily see the following.

Lemma 2.2: *For any bidirected graph $G = (V, A; \partial)$ and $w : V \to \mathbf{R}$ x is an optimal solution for $\min\{\sum_{v \in V} w(v)x(v) \,|\, x \in \mathrm{FD2P}(G)\}$ if and only if $2x - \mathbf{1}_V$ is an optimal solution for $\min\{\sum_{v \in V} w(v)x(v) \,|\, x \in \mathrm{IP}(G)\}$.* □

Note that the mapping $x \mapsto 2x - \mathbf{1}_V$ gives an isomorphism between two polytopes $\mathrm{FD2P}(G)$ and $\mathrm{IP}(G)$, that the half-integral points in $\mathrm{FD2P}(G)$ correspond to the ideals of G and that the integral points in $\mathrm{FD2P}(G)$ correspond to the spanning ideals of G.

The following theorem was implicitly given in [15, Proposition 3.1] and also follows from [1, Theorem 3.3]. However, we include the proof for the sake of completeness.

Theorem 2.3: *For any bidirected graph $G = (V, A; \partial)$ the ideal polytope $\mathrm{IP}(G)$ is integral.*
(Proof) Given a bidirected graph $G = (V, A; \partial)$ suppose that x is in $\mathrm{IP}(G)$. It suffices to show x is a convex combination of the characteristic vectors $\chi_{(X,Y)}$ ($(X, Y) \in \mathcal{I}(G)$).

Let the distinct positive values of $|x(v)|$ ($v \in V$) be given by

$$x_1 > x_2 > \cdots > x_p \,(> 0). \tag{2.10}$$

Define for each $i = 1, 2, \cdots, p$

$$U_i = \{v \,|\, v \in V, x(v) \geq x_i\},\ W_i = \{v \,|\, v \in V, -x(v) \geq x_i\} \tag{2.11}$$

Since $x \in \mathrm{IP}(G)$, we have $(U_i, W_i) \in \mathcal{I}(G)$ for each $i = 1, 2, \cdots, p$. Define λ_i ($i = 0, 1, \cdots, p$) by

$$\lambda_i = \begin{cases} 1 - x_1 & (i = 0) \\ x_i - x_{i+1} & (i = 1, \cdots, p-1) \\ x_p & (i = p) \end{cases} \tag{2.12}$$

Putting $(U_0, W_0) = (\emptyset, \emptyset)$ we have $x = \sum_{i=0}^{p} \lambda_i \chi_{(U_i, W_i)}$. Since $\sum_{i=0}^{p} \lambda_i = 1$ and $\lambda_i \geq 0$ ($i = 0, \cdots, p$), the present theorem follows. □

The following is a generalization of a result of Nemhauser and Trotter [13] for the fractional stable set polytopes.

Corollary 2.4: *For any bidirected graph $G = (V, A; \partial)$ the fractional degree-two polytope $\mathrm{FD2P}(G)$ is half-integral.*
(Proof) The present corollary follows from Proposition 2.1 and Theorem 2.3. □

Corollary 2.5: *For any bidirected graph G the linear programming problem over* FD2P(G) *can be reduced to the minimum-weight ideal problem for G, and vice versa.*
(Proof) The present corollary follows from Lemma 2.2 and Theorem 2.3 □

It follows from a result in [3] that a linear programming problem over a fractional degree-two polytope can be reduced to a minimum-cut problem.

3. Characterizations for a Fractional Degree-two Polytope to be Integral

In this section we argue conditions for a fractional degree-two polytopes to be integral. It is well-know that a necessary and sufficient condition for a fractional stable set polytope to be integral is that the given undirected graph is bipartite. Also, if the bidirected graph G is an ordinary directed graph (i.e., all the arcs of G are of type (3)), the FD2P(G) is the order ideal polytope and known tobe integral. These facts suggest the following generalization (Theorem 3.3) and is indeed true.

For any subset U of vertex set V the *reflection* of $G = (V, A; \partial)$ *by* U is the bidirected graph $G' = (V, A; \partial')$ defined as follows. For each arc $a \in A$, if $\partial a = \pm v \pm w$, we define

$$\partial' a = \pm\epsilon(v)v \pm \epsilon(w)w, \tag{3.1}$$

where for each $v \in V$

$$\epsilon(v) = \begin{cases} -1 & (\text{if } v \in U) \\ 1 & (\text{otherwise}). \end{cases} \tag{3.2}$$

We denote the reflection G' by $G{:}U$. See Figure 3.1.

For a bidirected graph $G = (V, A; \partial)$ an arc set $C \subseteq A$ is called *circle* if it forms a simple closed path of G, where we regard G as an undirected graph. A circle C is called *balanced* (*unbalanced*) if the number of arcs of type (1) in C plus the number of arcs of type (2) in C is even (odd). A bidirected graph is called *balanced* if it contains no unbalanced circle. The balancedness is characterized as follows.

Lemma 3.1 (Harary [8]): *A bidirected graph $G = (V, A; \partial)$ is balanced if and only if for some $U \subseteq V$ the reflection $G{:}U$ of G by U is an ordinary directed graph, i.e., all the arcs of $G{:}U$ are of type (3) (one head and one tail).* □

We have an another characterization of balanced bidirected graphs in terms of their ideals.

Lemma 3.2: *A bidirected graph $G = (V, A; \partial)$ is balanced if and only if there exists an orthant $(S, T) \in 3^V$ such that both of (S, T) and (T, S) are ideals of G.*
(Proof) Suppose that a bidirected graph $G = (V, A; \partial)$ is balanced. Then, by

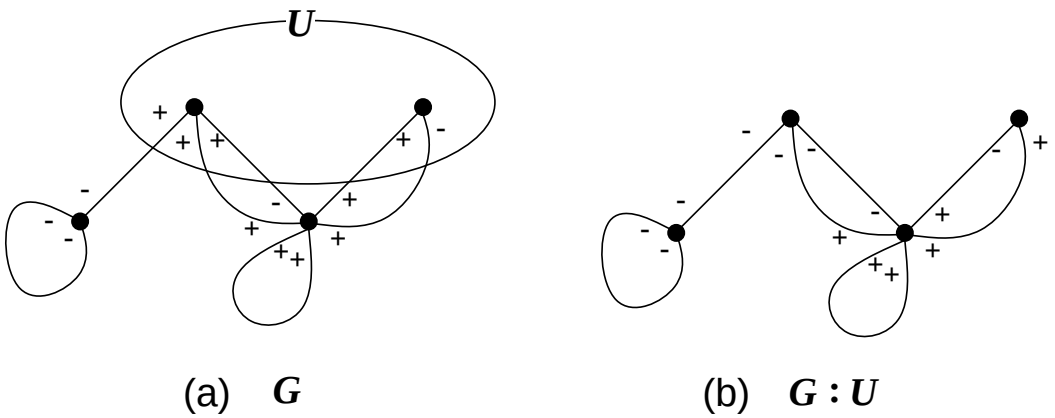

Figure 3.1: A reflection of a bidirected graph.

Lemma 3.1 there exists a subset U of V such that $G:U$ is an ordinary directed graph. Since both of $(V,\emptyset)$ and $(\emptyset,V)$ is ideals of $G:U$, it follows from the definitions that both of $(V-U,U)$ and $(U,V-U)$ are ideals of G.

Conversely, if $(V-U,U)$ and $(U,V-U)$ are ideals of G for some $U \subseteq V$, then $(V,\emptyset)$ and $(\emptyset,V)$ are ideals of $G:U$. This implies there is no arc of type (1) or of type (2) in $G:U$. It follows from Lemma 3.1 that G is balanced. □

Theorem 3.3: *The fractional degree-two polytope* FD2P(G) *associated with a bidirected graph* $G=(V,A;\partial)$ *is integral if and only if* G *is balanced.*
(Proof) The sufficiency easily follows from Lemma 3.1 since the left-hand side matrix of defining inequalities (2.8) is totally unimodular if (and only if) G is balanced.

To see the necessity, suppose that for a bidirected graph $G=(V,A;\partial)$ the associated FD2P(G) is integral. Then, it follows from Lemma 2.2 together with the remarks given after it that for any weight function there exists a minimum-weight ideal (S,T) such that $S \cup T = V$. Define a weight function $w: V \to \mathbf{R}$ on V by

$$w = -\sum_{a \in A} \partial a, \tag{3.3}$$

where we regard ∂a as a vector in $\mathbf{R}^V$. Then, for any ideal (X,Y) of G we have

$$\begin{aligned} w(X) - w(Y) &= \langle -\sum_{a \in A} \partial a, (X,Y) \rangle \\ &= -\sum_{a \in A} \langle \partial a, (X,Y) \rangle \\ &\geq 0. \end{aligned} \tag{3.4}$$

Let (S,T) be an optimal ideal for Problem $\min\{w(X)-w(Y)\,|\,(X,Y)\in\mathcal{I}(G)\}$ such that $S\cup T=V$. Since $(\emptyset,\emptyset)\in\mathcal{I}(G)$ and $w(\emptyset)-w(\emptyset)=0$, the optimal value is 0 and we have $w(S)-w(T)=0$. Then, we have $\langle\partial a,(S,T)\rangle=0$ for any $a\in A$. Hence, $\langle\partial a,(T,S)\rangle=0$ for any $a\in A$ and the signed subset (T,S) is also an (optimal) ideal of G. It follows from Lemma 3.2 that G is balanced. □

The following corollary is easily derived from Hoffman and Kruskal's theorem.

Corollary 3.4: *A fractional degree-two polytope* FD2P(G) *is integral if and only if for any integral* $l,u\in\mathbf{Z}^V$ *and* $c\in\mathbf{Z}^A$ *the polytope described by the system*

$$\langle\partial a,x\rangle\le c(a)\quad(a\in A),\tag{3.5}$$
$$l(v)\le x(v)\le u(v)\quad(v\in V)\tag{3.6}$$

of inequalities is integral. □

For any subset $Q\subseteq\mathbf{R}^V$ and any subset U of V we define the *reflection* $Q:U$ of Q by U by

$$Q{:}U=\{x{:}U\,|\,x\in Q\},\tag{3.7}$$

where $x{:}U$ is defined as

$$(x:U)(v)=\begin{cases}-x(v) & (\text{if } v\in U)\\ x(v) & (\text{otherwise})\end{cases}\qquad(v\in V).\tag{3.8}$$

Proposition 3.5: *Suppose that* $G=(V,A;\partial)$ *is a bidirected graph and* U *is a subset of* V. *We have* $\mathrm{IP}(G){:}U=\mathrm{IP}(G{:}U)$. □

For any $Q\subseteq[0,1]^V$ and $U\subseteq V$ define the *negation* $Q\,!\,U\subseteq[0,1]^V$ of Q at U by

$$Q\,!\,U=\{x\,!\,U\,|\,x\in Q\},\tag{3.9}$$

where $x\,!\,U$ is defined by

$$(x\,!\,U)(v)=\begin{cases}1-x(v) & (\text{if } v\in U)\\ x(v) & (\text{otherwise})\end{cases}\qquad(v\in V).\tag{3.10}$$

Corollary 3.6: *The fractional degree-two polytope* FD2P(G) *associated with a bidirected graph* $G=(V,A;\partial)$ *is integral if and only if* FD2P(G)$\,!\,U$ *is an order ideal polytope for some* $U\subseteq V$.

(Proof) Sufficiency is trivial. Suppose that the fractional degree-two polytope FD2P(G) is integral. Then, by Theorem 3.3, $G=(V,A;\partial)$ is balanced, i.e., for some $U\subseteq V$ the reflection $G{:}U$ of G is an ordinary directed graph and we have $\mathrm{IP}(G){:}U=\mathrm{IP}(G{:}U)$. It follows from Proposition 2.1 that $\mathrm{FD2P}(G)\,!\,U=\mathrm{FD2P}(G{:}U)$. Since FD2P($G{:}U$) is an order ideal polytope, the corollary holds. □

4. An Application

In this section we consider set systems characterized by "complete sets of implicants", which was developed by Hausmann and Korte [10] and Hausmann [9].

We call a pair $(V, \mathcal{F})$ of a nonempty finite set V and $\mathcal{F} \subseteq 2^V$ a *discrete system*. An *implicant* of $(V, \mathcal{F})$ is $(J^+, J^-) \in 3^V$ such that for any $F \subseteq V$ we have

$$J^+ \subseteq F, J^- \cap F = \emptyset \Longrightarrow F \notin \mathcal{F}. \tag{4.1}$$

The *length* of implicant (J^+, J^-) is $|J^+ \cup J^-|$. A set of implicants $\mathcal{J}$ with the property

$$F \notin \mathcal{F} \Longleftrightarrow \exists (J^+, J^-) \in \mathcal{J} : J^+ \subseteq F, J^- \cap F = \emptyset \tag{4.2}$$

is called a *complete set of implicants* of $(V, \mathcal{F})$.

If $\mathcal{J}$ is the complete set of implicants of a discrete system $(V, \mathcal{F})$ and all the implicants in $\mathcal{J}$ have lengths two, then we can associate a bidirected graph $G = (V, A; \partial)$ in an obvious way, i.e., A and ∂ are defined by

$$A = \{a_J \mid J = (J^+, J^-) \in \mathcal{J}\} \tag{4.3}$$

and

$$\partial a_J = \chi_J \quad (J = (J^+, J^-) \in \mathcal{J}) \tag{4.4}$$

and we see that $\mathcal{F}$ is the integral solutions of FD2P(G).

For $F_1, F_2 \in \mathcal{F}$ the *reduced system* with respect to F_1 and F_2 is the discrete system $(F_1 \Delta F_2, \tilde{\mathcal{F}})$, where Δ denotes the symmetric difference and

$$\tilde{\mathcal{F}} = \{F \mid F \subseteq F_1 \Delta F_2, F \cup (F_1 \cap F_2) \in \mathcal{F}\}. \tag{4.5}$$

For a discrete system $(V, \mathcal{F})$ define P($\mathcal{F}$) as the convex hull of the characteristic vectors χ_F ($F \in \mathcal{F}$).

Theorem 4.1 (Ikebe and Tamura [11]): *Suppose that $(V, \mathcal{F})$ is a discrete system and $F_1, F_2 \in \mathcal{F}$. If there exists a complete set of implicants $\tilde{\mathcal{J}}$ of the reduced system $(F_1 \Delta F_2, \tilde{\mathcal{F}})$ such that all the lengths of implicants in $\tilde{\mathcal{J}}$ are two, then* P($\tilde{\mathcal{F}}$) *is a negation of an order ideal polytope.*

(Proof) If $\tilde{\mathcal{J}}$ is the complete set of implicants of $(F_1 \Delta F_2, \tilde{\mathcal{F}})$ and all the implicants in $\tilde{\mathcal{J}}$ have lengths two, then we can associate a bidirected graph $G = (F_1 \Delta F_2, A; \partial)$ by (4.3) and (4.4), where $\mathcal{J}$ being replaced with $\tilde{\mathcal{J}}$. Then, $\tilde{\mathcal{F}}$ is the integral points of FD2P(G). Furthermore, since $F_1 \setminus F_2, F_2 \setminus F_1 \in \tilde{\mathcal{F}}$, the bidirected graph G must be balanced. It follows from Theorem 3.3 that FD2P(G) is integral, and hence, FD2P(G) = P($\tilde{\mathcal{F}}$). We complete the proof of the present theorem using Corollary 3.6. □

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References

[1] K. Ando and S. Fujishige: On structures of bisubmodular polyhedra. *Mathematical Programming* (to appear).

[2] K. Ando, S. Fujishige and T. Nemoto: Decomposition of a bidirected graph into strongly connected components and its signed poset structure. *Discrete Applied Mathematics* **68** (1996) 237-248.

[3] K. Ando, S. Fujishige and T. Nemoto: The minimum-weight ideal problem for signed posets. *Journal of Operations Research Society of Japan* (to appear).

[4] B. Aspvall, M. F. Plass and R. E. Tarjan: A linear-time algorithm for testing the truth of certain quantified Boolean formulas. *Information Processing Letters* **8** (1979) 121-123.

[5] J. Edmonds and E. L. Johnson: Matching: a well-solved class of linear programs. In: *Combinatorial Structures and Their Applications* (R. Guy, H. Hanani, N. Sauer and J. Schönheim, eds., Gordon and Breach, New York, 1970), pp. 88-92.

[6] P. L. Hammer and S. Nguyen: A partial order in the solution space of bivalent programs. In: *Combinatorial Optimization* (N. Christofides, A. Mingozzi, P. Toth and C. Sandi, eds., John Wiley & Sons, 1979), pp. 93-106.

[7] P. Hansen and B. Jaumard: Uniquely solvable quadratic Boolean equations. *Discrete Applied Mathematics* **12** (1985) 147-154.

[8] F. Harary: On the notion of balance of a signed graph. *Michigan Mathematical Journal* **2** (1953-1954) 143-146.

[9] D. Hausmann: *Adjacency on Polytopes in Combinatorial Optimizations* (Oelgeschlager, Gunn & Hain, Cambridge, 1979).

[10] D. Hausmann and B. Korte: Colouring criteria for adjacency on 0-1 polyhedra. *Mathematical Programming Study* **8** (1978) 106-127.

[11] Y. T. Ikebe and A. Tamura: Ideal polytopes and face structures of some combinatorial optimization problems. *Mathematical Programming* **71** (1995) 1-16.

[12] E. L. Johnson and M. W. Padberg: Degree-two inequalities, clique facets, and biperfect graphs. *Annals of Discrete Mathematics* **16** (1982) 169-187.

[13] G. L. Nemhauser and L. E. Trotter Jr.: Vertex packings: structural properties and algorithms. *Mathematical Programming* **8** (1975) 232-248.

[14] S. Rudeanu: Boolean functions and generalized minterms. *Discrete Applied Mathematics* **49** (1983) 241-248.

[15] V. Reiner: Signed posets. *Journal of Combinatorial Theory*, Ser. A **62** (1993) 324-360.

[16] T. Zaslavsky: Orientation of signed graphs. *European Journal of Combinatorics* **12** (1991) 361-375.