Extreme Point Axioms for Closure Spaces

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Koshevoy's Theorem

Theorem 1 (Koshevoy (1999)): *A mapping $S\colon 2^X \to 2^X$ is the extreme point operator of an antimatroid if and only if S satisfies*

(Ex1) $S(A) \subseteq A \quad (A \subseteq X)$. (Intensionality)

(NE) $S(A) \neq \emptyset \quad (\emptyset \neq A \subseteq X)$. (Nonemptiness)

(PI) $\forall A, B \subseteq X\colon S(A \cup B) = S(S(A) \cup S(B))$.
(path-independence)

Results

- Characterization of the extreme point operators of closure spaces

- Characterization of the extreme point operators of matroids

$\Longrightarrow$ New axiom systems for closure spaces and matroids

Closure Spaces

X finite set. Closure operator $\tau\colon 2^X \to 2^X$, i.e.,

(C1) $\forall A \subseteq X\colon A \subseteq \tau(A)$. (Extensionality)

(C2) $\forall A, B \subseteq X\colon A \subseteq B \Longrightarrow \tau(A) \subseteq \tau(B)$. (Monotonicity)

(C3) $\forall A \subseteq X\colon \tau(\tau(A)) = \tau(A)$. (Idempotence)

We call (X, τ) a **closure space**.

Cryptomorphism

(i) Moore families

(ii) Complete Implicational Systems ([Armstrong])

(iii) Finite lattices

(iv) Overhanging relations [Domenach and Leclerc (2002)]

Moore Families

A family $\mathcal{L} \subseteq 2^X$ is called a **Moore family** if

(A1) $X \in \mathcal{L}$,

(A2) $A, B \in \mathcal{L} \Longrightarrow A \cap B \in \mathcal{L}$.

- $\mathcal{L}$ ordered by inclusion $\subseteq$ forms a lattice.

- Every finite lattice is isomorphic to a Moore family.

Moore Families $\Leftrightarrow$ Closure Spaces

For a closure space (X, τ), $\mathcal{L} \subseteq 2^X$ defined by

$$\mathcal{L} = \{A \mid A \in 2^X, \tau(A) = A\}.$$

is a Moore family. We call a member in $\mathcal{L}$ **closed**.

Given a Moore family $\mathcal{L} \subseteq 2^X$, $\tau_{\mathcal{L}}\colon 2^X \to 2^X$ defined by

$$\tau_{\mathcal{L}}(A) = \bigcap\{C \mid A \subseteq C \in \mathcal{L}\} \quad (A \subseteq X).$$

is a closure operator. The closed subsets of $(X, \tau_{\mathcal{L}})$ are given by $\mathcal{L}$.

Subclasses of Closure Spaces

Closure space (X, τ) is a **matroid** if

(EA) $\forall A \subseteq X, \forall p, q \notin \tau(A)$: $q \in \tau(A \cup p) \Longrightarrow p \in \tau(A \cup q)$.

(Steinitz-MacLane Exchange Axiom)

Closure space (X, τ) is an **antimatroid** (or convex geometry) if

(C0) $\tau(\emptyset) = \emptyset$.

(AE) $\forall A \subseteq X, \forall p, q \notin \tau(A)$ with $p \neq q$:

$$q \in \tau(A \cup p) \Longrightarrow p \notin \tau(A \cup q).$$

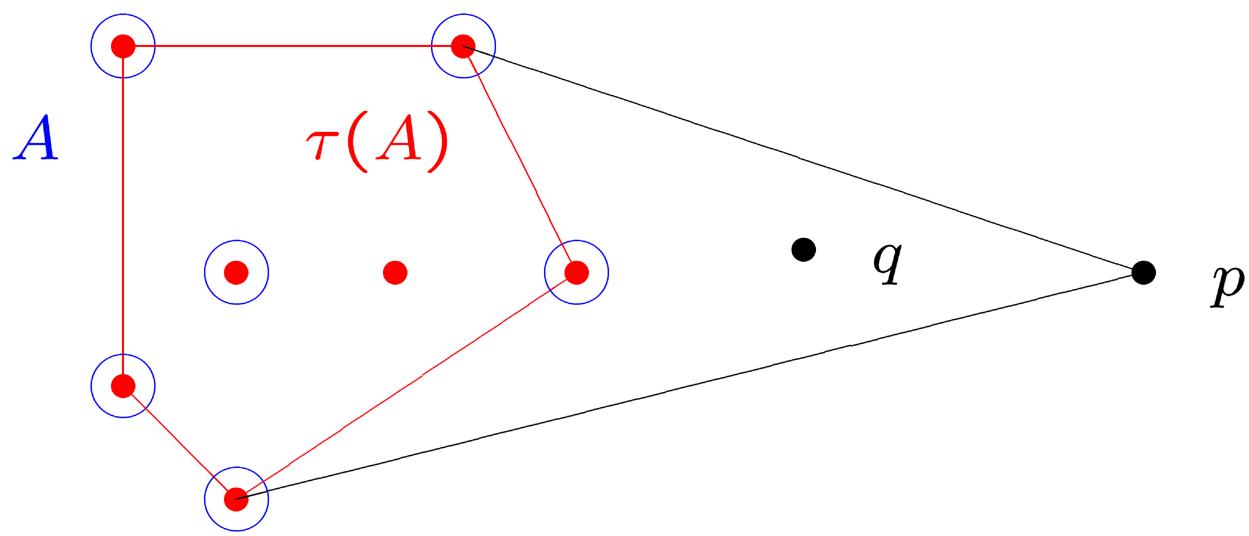
(Antiexchange Axiom)

Example of antimatroid 1 – convex shelling –

For a finite set $X \subseteq \mathbf{R}^n$, define $\tau\colon 2^X \to 2^X$ by

$$\tau(A) = \text{conv.hull}(A) \cap X \quad (A \subseteq X).$$

Then, (X, τ) is an antimatroid.


A
τ(A)
q
p

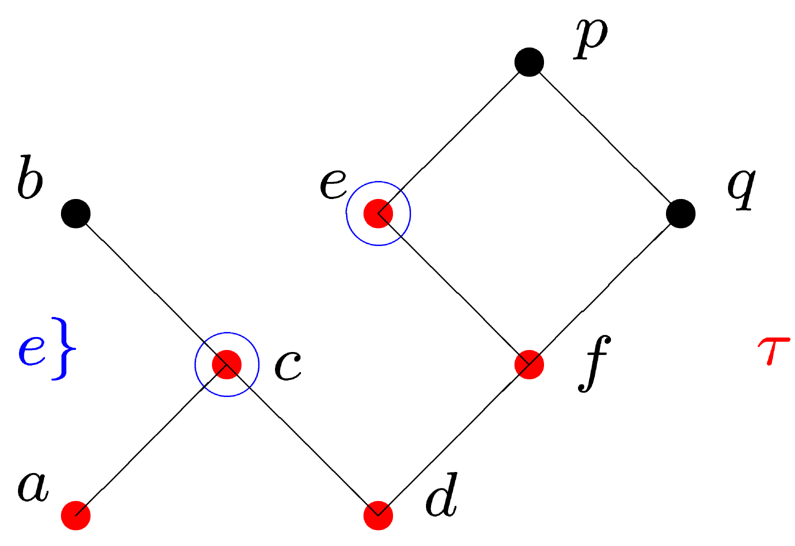
Example of antimatroid 2 – poset shelling –

For a finite poset $P = (X, \preceq)$, define $\tau\colon 2^X \to 2^X$ by

$\tau(A) = \{x | x \in X, \exists a \in A\colon x \preceq a\}$ = (the ideal generated by A)

Then (X, τ) is an antimatroid.

$A = \{c, e\}$



$\tau(A) = \{a, c, d, e, f\}$

Extreme point operator ex

Let (X, τ) be a closure space. Define ex: $2^X \to 2^X$ by

$$\text{ex}(A) = \{p \mid p \in A, p \notin \tau(A - p)\} \quad (A \subseteq X).$$

We call ex the **extreme point operator** of (X, τ). $p \in \text{ex}(A)$ is an **extreme point** of A.

Example:

- If (X, τ) is a matroid, ex(A) = the isthmuses of A.
- For a convex shelling (X, τ), ex(A) = the extreme points of conv$(A) \subseteq \mathbf{R}^X$.
- For a poset shelling (X, τ), ex(A) = the maximal elements in A.

We may have ex$(A) = \emptyset$ for $A \neq \emptyset$.

Properties of Extreme Point Operators

Proposition 2: *Let* (X, τ) *be a closure space. Then,* ex: $2^X \to 2^X$ *satisfies the following* (Ex1)–(Ex3).

(Ex1) $\forall A \subseteq X$: $\text{ex}(A) \subseteq A$. (Intensionality)

(Ex2) $A \subseteq B \subseteq X \Longrightarrow \text{ex}(B) \cap A \subseteq \text{ex}(A)$. (Chernoff property)

(Ex3) $\forall A \subseteq X, \forall p, q \notin A$:

$$(p \notin \text{ex}(A \cup p), q \in \text{ex}(A \cup q)) \Longrightarrow q \in \text{ex}(A \cup p \cup q).$$

(Proof)

(Ex1) $\forall A \subseteq X$: ex$(A) \subseteq A$ is clear.

(Ex2) $A \subseteq B \subseteq X \Longrightarrow$ ex$(B) \cap A \subseteq$ ex(A).

Let $p \in$ ex$(B) \cap A$. Then, $p \notin \tau(B - p)$. Since $\tau(A - p) \subseteq \tau(B - p)$, we have $p \notin \tau(A - p)$. Hence $p \in$ ex(A).

(Ex3) $\forall A \subseteq X, \forall p, q \notin A$:

$(p \notin \mathrm{ex}(A \cup p), q \in \mathrm{ex}(A \cup q)) \Longrightarrow q \in \mathrm{ex}(A \cup p \cup q)$.

$$\begin{aligned} \text{(LHS)} &\Longrightarrow p \in \tau(A), q \notin \tau(A) \\ &\Longrightarrow \tau(A \cup p) = \tau(A) \not\ni q \\ &\Longrightarrow p \in \mathrm{ex}(A \cup p \cup q). \end{aligned}$$

□

Axiom System by Extreme Point Operator

Theorem 3 (A): *A mapping* $S\colon 2^X \to 2^X$ *is the extreme point operator of a closure space if and only if it satisfies*

(Ex1) $\forall A \subseteq X\colon S(A) \subseteq A.$ (Intensionality)

(Ex2) $A \subseteq B \subseteq X \Longrightarrow S(B) \cap A \subseteq S(A).$ (Chernoff property)

(Ex3) $\forall A \subseteq X, \forall p, q \notin A$:

$$(p \notin S(A \cup p), q \in S(A \cup q)) \Longrightarrow q \in S(A \cup p \cup q). \ \square$$

Theorem 4: *Suppose that* $S: 2^X \to 2^X$ *satisfies* (Ex1)–(Ex3). *Define* $\tau: 2^X \to 2^X$ *by*

$$\tau_S(A) = A \cup \{p \mid p \notin A, p \notin S(A \cup p)\}. \qquad (1)$$

Then, (X, τ_S) *is a closure space with its extreme point operator being* S. □

Axiom System for Matroids

Theorem 5 (A): *A mapping* $S: 2^X \to 2^X$ *is the extreme point operator of a matroid if and only if it satisfies (Ex1)∼(Ex3) and*

(Ex4) $\forall A \subseteq X, \forall p \in X: p \in S(A \cup p) \Rightarrow S(A \cup p) \supseteq S(A) \cup p$.

□

Axiom System for Antimatroids

Theorem 6: *A mapping $S: 2^X \to 2^X$ is the extreme point operator of an antimatroid if and only if it satisfies*

(Ex0) $\forall p \in X: S(\{p\}) = \{p\}$,

(Ex1) $\forall A \subseteq X: S(A) \subseteq A$,

(Ex2) $A \subseteq B \subseteq X \Rightarrow S(B) \cap A \subseteq S(A)$, *and*

(Ex5) $\forall A, B \subseteq X: S(B) \subseteq A \subseteq B \Rightarrow S(A) \subseteq S(B)$.

(Aizerman's Axiom) □

Concluding Remarks

- Characterization of independence families of closure spaces (or antimatroid),
  (where $A \subseteq X$ is independent $\Leftrightarrow$ ex(A) $= A$.)

- We may obtain general result replacing path-independent choice functions with functions satisfying conditions (Ex1)-(Ex3).