Extreme Point Characterizations of Closure Spaces

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Koshevoy's Work

A choice function on X is a mapping $S\colon 2^X \to 2^X$ satisfying the following two conditions.

(Ex1) $S(A) \subseteq A \quad (A \subseteq X)$. (Intensionality)

(NE) $S(A) \neq \emptyset \quad (\emptyset \neq A \subseteq X)$. (Nonemptiness)

Theorem 0 [Koshevoy]: A mapping $S\colon 2^X \to 2^X$ is the extreme point operator of an antimatroid if and only if S satisfies (Ex1), (NE) and the following (PI).

(PI) $\forall A, B \subseteq X\colon S(A \cup B) = S(S(A) \cup S(B))$. (Path Independence)

$\square$

Antimatroids (Convex geometries)

Let X be a finite set and $\tau\colon 2^X \to 2^X$. The pair (X, τ) is an antimatroid (or convex geometry) if

(C0) $\tau(\emptyset) = \emptyset$.

(C1) $\forall A \subseteq X\colon A \subseteq \tau(A)$. (Extensionality)

(C2) $\forall A, B \subseteq X\colon A \subseteq B \Longrightarrow \tau(A) \subseteq \tau(B)$. (Monotonicity)

(C3) $\forall A \subseteq X\colon \tau(\tau(A)) = \tau(A)$. (Idempotence)

(AE) $\forall A \subseteq X, \forall p, q \notin \tau(A)$ with $p \neq q$:

$$q \in \tau(A \cup p) \Longrightarrow p \notin \tau(A \cup q).$$

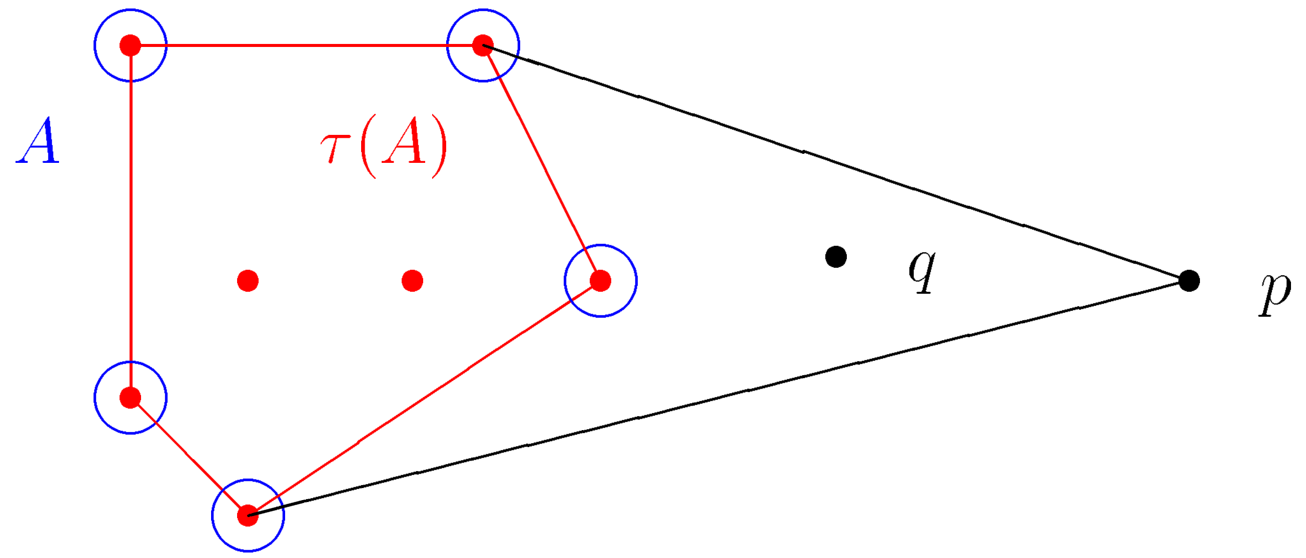
(Antiexchange Axiom)

Example of antimatroid 1 – convex shelling –

For a finite set $X \subseteq \mathbf{R}^n$, define $\tau\colon 2^X \to 2^X$ by

$$\tau(A) = \text{conv.hull}(A) \cap X \quad (A \subseteq X).$$

Then, (X, τ) is an antimatroid.



$$\forall A \subseteq X, \forall p, q \notin \tau(A)\colon q \in \tau(A \cup p) \Longrightarrow p \notin \tau(A \cup q).$$

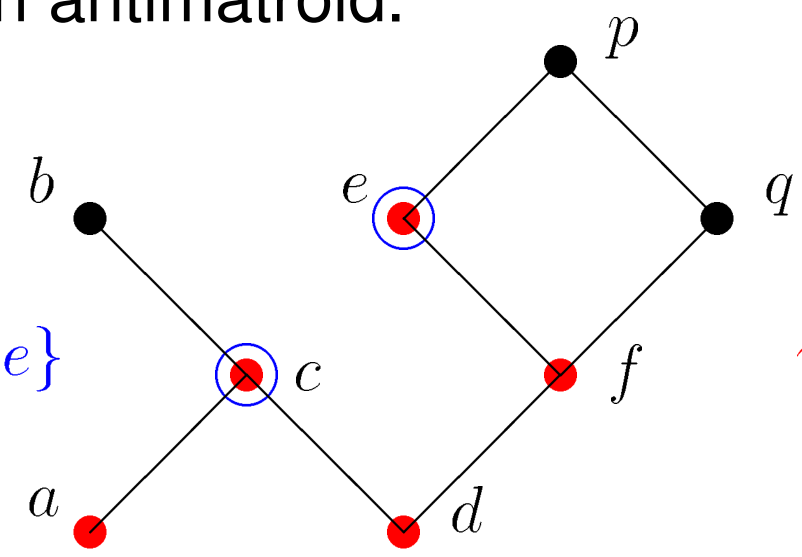
Example of antimatroid 2 – poset shelling –

For a finite poset $P = (X, \preceq)$, define $\tau\colon 2^X \to 2^X$ by

$\tau(A) = \{x | x \in X, \exists a \in A\colon x \preceq a\} =$ (the ideal generated by A).

Then (X, τ) is an antimatroid.

$A = \{c, e\}$



$\tau(A) = \{a, c, d, e, f\}$

$$\forall A \subseteq X, \forall p, q \notin \tau(A)\colon q \in \tau(A \cup p) \Longrightarrow p \notin \tau(A \cup q).$$

Extreme point operator ex

Let (X, τ) be an antimatroid.
Define $\mathrm{ex}\colon 2^X \to 2^X$ by

$$\mathrm{ex}(A) = \{p \mid p \in A, p \notin \tau(A - p)\} \quad (A \subseteq X).$$

We call ex the extreme point operator of (X, τ). $p \in \mathrm{ex}(A)$ is an extreme point of A.
Example:

- For a convex shelling (X, τ), $\mathrm{ex}(A)$ = the extreme points of $\mathrm{conv}(A) \subseteq \mathbf{R}^X$.
- For a poset shelling (X, τ), $\mathrm{ex}(A)$ = the maximal elements in A.

Closure Spaces

X finite set. Closure operator $\tau\colon 2^X \to 2^X$, i.e.,

(C1) $\forall A \subseteq X\colon A \subseteq \tau(A)$. (Extensionality)

(C2) $\forall A, B \subseteq X\colon A \subseteq B \Longrightarrow \tau(A) \subseteq \tau(B)$. (Monotonicity)

(C3) $\forall A \subseteq X\colon \tau(\tau(A)) = \tau(A)$. (Idempotence)

We call (X, τ) a closure space.

Matroids are the closure space satisfying Exchange Axiom and antimatroids are the closure spaces satisfying Antiexchange Axiom and $\tau(\emptyset) = \emptyset$.

Cryptomorphic Axiom Systems

- Moore family (defined later)
- Armstrong's Axiom for functional dependency
- Finite lattice
- Extreme Point Operator [A (2002)]

Closed sets

Proposition 1: For a closure space (X,τ), define $\mathcal{L} \subseteq 2^X$ by

$$\mathcal{L} = \{A \mid A \in 2^X, \tau(A) = A\}.$$

Then, we have

(A1) $X \in \mathcal{L}$.

(A2) $A, B \in \mathcal{L} \Longrightarrow A \cap B \in \mathcal{L}$. $\Box$

We call a member in $\mathcal{L}$ closed.
Example:

- For a convex shelling (X,τ), the closed sets = "convex set".
- For a poset shelling (X,τ), the closed sets = the ideals.

Conversely, given a family $\mathcal{L} \subseteq 2^X$ satisfying

(A1) $X \in \mathcal{L}$.

(A2) $A, B \in \mathcal{L} \Longrightarrow A \cap B \in \mathcal{L}$,

define $\tau_{\mathcal{L}}\colon 2^X \to 2^X$ by

$$\begin{aligned} \tau_{\mathcal{L}}(A) &= \bigcap\{C \mid A \subseteq C \in \mathcal{L}\} \\ &= \text{the minimum member of } \mathcal{L} \text{ containing } A \quad (A \subseteq X). \end{aligned}$$

Proposition 2: $\tau_{\mathcal{L}}$ is a closure operator, i.e., $(X, \tau_{\mathcal{L}})$ a closure space. Moreover, the closed subsets are given by $\mathcal{L}$. $\square$

A family of subsets of X satisfying (A1) and (A2) is called Moore family.

Extreme point operator ex

Let (X, τ) be a closure space.
Define $\mathrm{ex}\colon 2^X \to 2^X$ by

$$\mathrm{ex}(A) = \{p \mid p \in A, p \notin \tau(A - p)\} \quad (A \subseteq X).$$

We call ex the extreme point operator of (X, τ). $p \in \mathrm{ex}(A)$ is an extreme point of A.

We may have $\mathrm{ex}(A) = \emptyset$ for $A \neq \emptyset$.

Free subsets

Let (X, τ) be a closure space.
$A \subseteq X$ is called free if $\mathrm{ex}(A) = A$.
Example:

- For a convex shelling (X, τ), free sets = extreme.
- For a poset shelling (X, τ), A is free $\leftrightarrow$ A is an antichain.

Properties of Extreme Point Operators

Proposition 3: [Chernoff Property] $A \subseteq B \subseteq X$ implies $\mathrm{ex}(B) \cap A \subseteq \mathrm{ex}(A)$. $\square$

Lemma 4: Let (X, τ) be a closure space. For each $A \subseteq X$, we have

$$\mathrm{ex}(A) = \bigcap \{B \mid B \subseteq A, \tau(B) = \tau(A)\}.$$

$\square$

A Characterization of Closure Space

Theorem 5: A mapping $S\colon 2^X \to 2^X$ is the extreme point operator of a closure space if and only if it satisfies

(Ex1) $\forall A \subseteq X\colon S(A) \subseteq A$. (Intensionality)

(Ex2) $A \subseteq B \subseteq X \Longrightarrow S(B) \cap A \subseteq S(A)$. (Chernoff property)

(Ex3) $\forall A \subseteq X, \forall p, q \notin A$:
$(p \notin S(A \cup p), q \in S(A \cup q)) \Longrightarrow q \in S(A \cup p \cup q)$.

$\square$

(sketch of) the proof

(The "only if" part) $\Rightarrow$ [BB]

(The "if" part)
Theorem 6: Suppose that $S\colon 2^X \to 2^X$ satisfies (Ex1)–(Ex3).
Define $\tau\colon 2^X \to 2^X$ by

$$\tau_S(A) = A \cup \{p \mid p \notin A, p \notin S(A \cup p)\}.$$

Then, (X, τ_S) is a closure space with its extreme point operator being S. $\square$

Koshevoy's Work

Theorem 0: [Koshevoy] A mapping $S\colon 2^X \to 2^X$ is the extreme point operator of an antimatroid if and only if S satisfies

(NE) $S(A) \neq \emptyset \quad (\emptyset \neq A \subseteq X)$. (Nonemptiness)

(Ex1) $S(A) \subseteq A \quad (A \subseteq X)$. (Intensionality)

(PI) $\forall A, B \subseteq X\colon S(A \cup B) = S(S(A) \cup S(B))$. (Path Independence)

$\square$

A Characterization of Antimatroids

Proposition 7: A mapping $S\colon 2^X \to 2^X$ is the extreme point operator of an antimatroid if and only if it satisfies

(Ex0) $\forall p \in X\colon S(\{p\}) = \{p\}$. (Singleton Identity)

(Ex1) $\forall A \subseteq X\colon S(A) \subseteq A$.

(Ex2) $A \subseteq B \subseteq X \Rightarrow S(B) \cap A \subseteq S(A)$

and

(Ex5) $\forall A, B \subseteq X\colon S(B) \subseteq A \subseteq B \Rightarrow S(A) \subseteq S(B)$. (Aizerman's Axiom)

$\square$

Theorem 8 [Aizerman and Malishevski]: Condition (PI) is equivalent to Conditions (Ex2) and (Ex5), provided that (Ex1) holds. □

Theorem 9: Condition (Ex5) implies Condition (Ex3), provided that Conditions (Ex1)–(Ex2) hold. □

Koshevoy Map

$$\bar{S}(A) = \bigcup\{B \mid B \subseteq X, S(B) = S(A)\} \quad (A \subseteq X).$$

$$\tau_S(A) = A \cup \{p \mid p \notin A, p \notin S(A \cup p)\}.$$

Proposition 10: Suppose that mapping $S\colon 2^X \to 2^X$ satisfies Conditions (Ex0)–(Ex2) and (Ex5). Then, we have $\tau_S = \bar{S}$. $\Box$

Matroids

Let X be a finite set and $\tau\colon 2^X \to 2^X$. (X, τ) is called a matroid if

(C1) $\forall A \subseteq X\colon A \subseteq \tau(A)$. (Extensionality)

(C2) $\forall A, B \subseteq X\colon A \subseteq B \Longrightarrow \tau(A) \subseteq \tau(B)$. (Monotonicity)

(C3) $\forall A \subseteq X\colon \tau(\tau(A)) = \tau(A)$. (Idempotence)

(EA) $\forall A \subseteq X, \forall p, q \notin \tau(A)\colon q \in \tau(A \cup p) \Longrightarrow p \in \tau(A \cup q)$.
(Steinitz-MacLane Exchange Axiom)

Example of matroid – linear matroid –

For a finite set $X \subseteq \mathbf{R}^n$, define $\tau\colon 2^X \to 2^X$ by

$$\begin{aligned}\tau(A) &= \text{lin.hull}(A) \cap X \\ &= \{x \in X \mid x \text{ is linearly dependent to } A\} \quad (A \subseteq X).\end{aligned}$$

Then, (X, τ) is a matroid.

$\forall A \subseteq X, \forall p, q \notin \tau(A)\colon q \in \tau(A \cup p) \Longrightarrow p \in \tau(A \cup q)$.

A Characterization of Matroids

Theorem 11: A mapping $S\colon 2^X \to 2^X$ is the extreme point operator of a matroid if and only if it satisfies

(Ex1) $\forall A \subseteq X\colon S(A) \subseteq A$.

(Ex2) $A \subseteq B \subseteq X \Rightarrow S(B) \cap A \subseteq S(A)$.

(Ex3) $\forall A \subseteq X, \forall p, q \notin A$:

$$(p \notin S(A \cup p), q \in S(A \cup q)) \Longrightarrow q \in S(A \cup p \cup q).$$

and

(Ex4) $\forall A \subseteq X, \forall p \in X\colon p \in S(A \cup p) \Rightarrow S(A \cup p) \supseteq S(A) \cup p$.

□