Faigle-Kern の双対貪欲算法とその帰結

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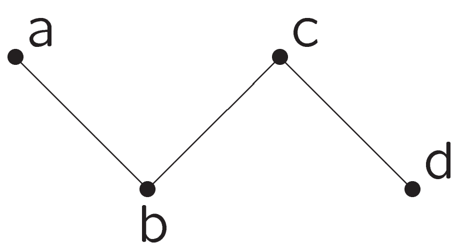
Outline of this Talk

1. Review of submodular system
2. Review of submodular functions on antichains and Dual Greedy Algorithm
3. Lovász extension
4. An algorithm for testing a vector for being an extreme point
5. Concluding Remarks

Submodular Systems

- $P = (E, \preceq)$: partially ordered set (poset). ($n = |E|$.)



- $I \subseteq E$ is an ideal of $P \Leftrightarrow (y \preceq x \in I \Rightarrow y \in I)$.
  $\mathcal{I}(P) =$ the set of ideals of P.

Remark: $\mathcal{I}(P)$ *forms a distributive lattice with lattice operations* $\cup$ *and* $\cap$.

- $f\colon \mathcal{I}(P) \to \mathbf{R}$ is *submodular* $\Leftrightarrow$

$$\forall A, B \in \mathcal{I}(P)\colon f(A) + f(B) \geq f(A \cup B) + f(A \cap B).$$

- Validity of Greedy Algorithm for solving

$$\left|\begin{array}{ll} \text{max} & \sum\{w(e)x(e) \mid e \in E\} \\ \text{s.t.} & x(X) \le f(X) \qquad\qquad (X \in \mathcal{I}(P)), \end{array}\right.$$

  where $x(A) = \sum_{e \in A} x(e)$.

- $f\colon \mathcal{I}(P) \to \mathbf{R}$ is submodular iff its Lovász extension is convex.

- Existence of polynomial time algorithm for minimizing f.

- TDIity of $x(X) \le f(X)$ $(X \in \mathcal{I}(P))$.

- Intersection Theorem

- ...

Submodular Functions on Antichains

- $A \subseteq E$ is an *antichain* of $P = (E, \preceq)$ if $\forall a, b \in A \colon a \preceq b \Rightarrow a = b$.

- $\mathcal{A}(P)$ the set of antichains of P.

- For $f \colon \mathcal{A}(P) \to \mathbf{R}$ define

$$\mathrm{P}(f) = \{x \mid x \in \mathbf{R}^E, \forall A \in \mathcal{A}(P) \colon x(A) \le f(A)\}.$$

Validity of Dual Greedy Algorithm

$$
\text{(P)}\left|\begin{array}{lll}
\text{max} & \sum\{w(e)x(e) \mid e \in E\} & \\
\text{s.t.} & x(A) \le f(A) & \qquad (A \in \mathcal{A}(P))
\end{array}\right.
$$

- P: rooted forest + $f\colon \mathcal{A}(P) \to \mathbf{R}$: submodular [Faigle and Kern (1996)]

- $f\colon \mathcal{A}(P) \to \mathbf{R}$: submodular and monotone [Faigle and Kern (2000)]

- $f\colon \mathcal{A}(P) \to \mathbf{R}$: b-submodular [Krüger (2000)]

Ideals and Antichains

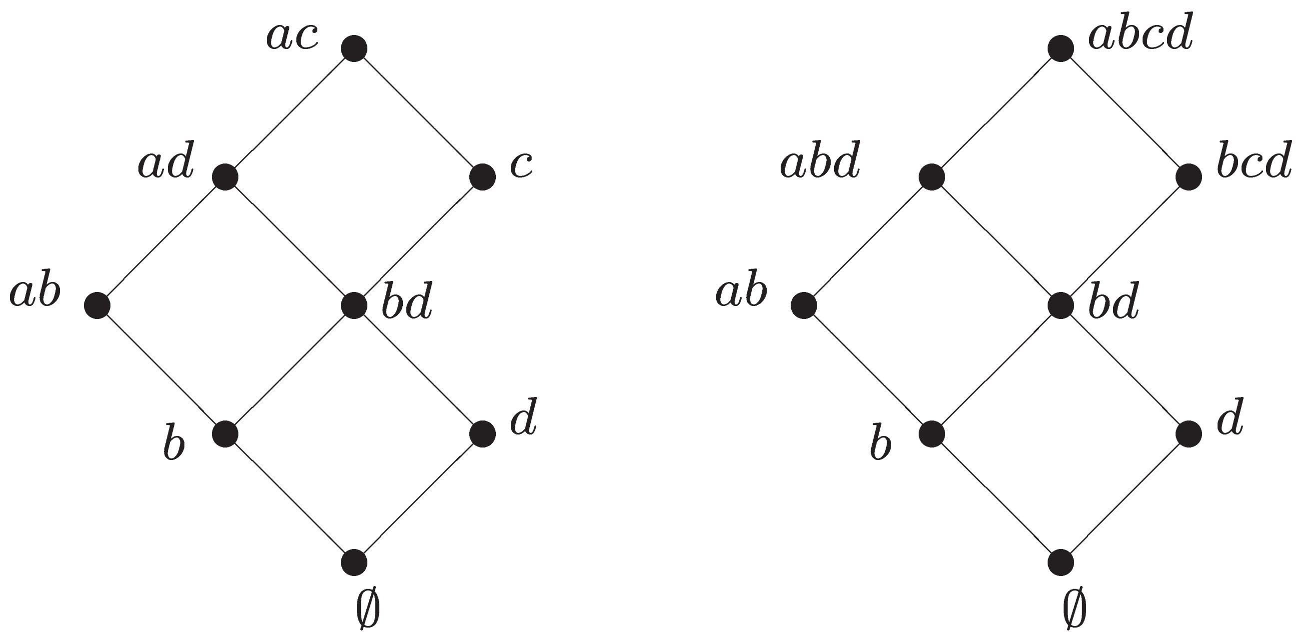
- The ideal generated by $A \in \mathcal{A}(P)$

$$\mathrm{id}(A) = \{y \mid y \in E, \exists x \in A\colon y \preceq x\} \qquad \in \mathcal{I}(P).$$

- For $I \in \mathcal{I}(P)$,

$$\max(I) = \{x \mid x \in I, \nexists y \in I\colon x \prec y\} \qquad \in \mathcal{A}(P).$$

Proposition: *The mappings* $\mathrm{id}\colon \mathcal{A}(P) \to \mathcal{I}(P)$ *and* $\max\colon \mathcal{I}(P) \to \mathcal{A}(P)$ *are one-to-one correspondence. Each of them is the inverse of the other.* $\square$



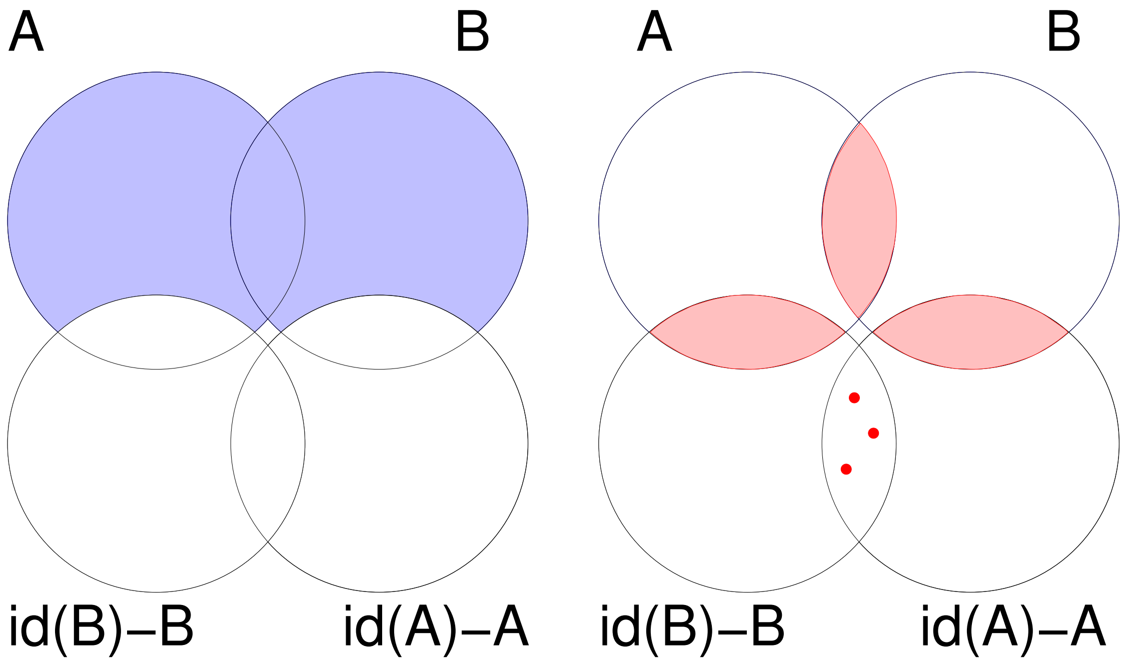
$$\mathcal{A}(P) \underset{\max}{\overset{\mathrm{id}}{\rightleftarrows}} \mathcal{I}(P)$$

Corollary: *The bijections induce lattice operations on* $\mathcal{A}(P)$*:* $(\mathcal{I}(P); \cup, \cap) \cong (\mathcal{A}(P); \vee, \wedge)$, *where*

$$\begin{aligned} A \vee B &= \max(\mathrm{id}(A) \cup \mathrm{id}(B)), \\ A \wedge B &= \max(\mathrm{id}(A) \cap \mathrm{id}(B)). \end{aligned}$$

and partial order $\preceq$ on $\mathcal{A}(P)$ as

$$\forall A, B \in \mathcal{A}(P)\colon A \preceq B \;\Leftrightarrow\; \mathrm{id}(A) \subseteq \mathrm{id}(B). \;\square$$



$A \vee B$ $A \wedge B$

Dual Greedy Algorithm

- $f\colon \mathcal{A}(P) \to \mathbf{R}$ is *submodular* $\Leftrightarrow$ $A, B \in \mathcal{A}(P)$:
$$f(A) + f(B) \geq f(A \vee B) + f(A \wedge B).$$

- The polyhedron associated with $f\colon \mathcal{A}(P) \to \mathbf{R}$
$$\mathsf{P}(f) = \{x \mid x \in \mathbf{R}^E, \forall A \in \mathcal{A}(P)\colon x(A) \leq f(A)\}.$$

- LP over $\mathsf{P}(f)$ and its dual

$$\text{(P)}\;\left|\;\begin{array}{lll} \max & \sum\{w(e)x(e) \mid e \in E\} & \\ \text{s.t.} & x(A) \leq f(A) & (A \in \mathcal{A}(P)) \end{array}\right.$$

$$\text{(D)}\;\left|\;\begin{array}{lll} \min & \sum\{f(A)y(A) \mid A \in \mathcal{A}(P)\} & \\ \text{s.t.} & \sum\{y(A) \mid e \in A \in \mathcal{A}(P)\} = w(e) & (e \in E), \\ & y(A) \geq 0 & (A \in \mathcal{A}(P)) \end{array}\right.$$

$$
\left(
\begin{array}{c c}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_n \end{array} \\
\begin{array}{c} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_n \end{array} &
\begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
* & 1 & 0 & \cdots & 0 \\
* & * & 1 & \ddots & \vdots \\
\vdots & \cdots & \cdots & \ddots & 0 \\
* & * & \cdots & \cdots & 1
\end{bmatrix}
\begin{bmatrix} x(e_1) \\ x(e_2) \\ \vdots \\ \vdots \\ x(e_n) \end{bmatrix}
=
\begin{bmatrix} f(A_1) \\ f(A_2) \\ \vdots \\ \vdots \\ f(A_n) \end{bmatrix}
\end{array}
\right)
$$

where each row is the characteristic vector χ_{A_i} of A_i.

For $X \subseteq E$ define the characteristic vector $\chi_X : E \to \{0, 1\}$ by

$$\chi_X(e) = \begin{cases} 1 & \text{if } e \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma: *y^* is feasible for (D).* □

Algorithm 1 Dual Greedy (Faigle and Kern(1996))

```
y* ← 0.
for i = n downto 1 do
    Choose e_i ∈ max(E) such that
    w(e_i) = min{w(e) | e ∈ max(E)}.
    A_i ← max E.
    y*(A_i) ← w(e_i).
    w(e) ← w(e) − w(e_i)    (e ∈ max(E)).
    {w(e) is kept nonnegative and so is y*.}
    E ← E − {e_i}.
end for
Define x* as the unique solution of the system

        x(A_i) = f(A_i)    (i = 1, ···, n)

of equations.
{x* and y* satisfies complementary slackness.}
```

Theorem (Faigle and Kern 1996): *If* $P = (E, \preceq)$ *is a rooted forest and* $f\colon \mathcal{A}(P) \to \mathbf{R}_+$, *then* $x^* \in$ P(f) *is feasible for (P), and hece, Dual Greedy Algorithm is valid for any* $w\colon E \to \mathbf{R}_+$.

Theorem (Faigle and Kern 2000): *Let* $P = (E, \preceq)$ *be an arbitrary poset and* f *is submodular and monotone, then Dual Greedy Algorithm is valid for any* $w\colon E \to \mathbf{R}_+$.

Submodularity in Krüger's sense

- $f\colon \mathcal{A}(P) \to \mathbf{R}$ が b-submodular

$\Leftrightarrow$ $\forall A, B \in \mathcal{A}(P)$:

$$f(A) + f(B) \geq f(A \vee B) + f(A \sqcap B),$$

where

$$A \sqcap B = (A \wedge B) \cap (A \cup B).$$

Remark: *If P is a rooted forest, we have $A \wedge B = A \sqcap B$, and hence, submodularity and b-submodularity are equivalent.*

Theorem (Krüger 2000): *For any poset P, if $f\colon \mathcal{A}(P) \to \mathbf{R}$ b-submodular, Dual Greedy Algorithm is valid for each $w\colon E \to \mathbf{R}_+$.* □

Corollary: *Let P be an arbitrary poset. For a function $f\colon \mathcal{A}(P) \to \mathbf{R}$ Dual Greedy Algorithm is valid for each $w\colon E \to \mathbf{R}_+$ if and only if f is b-submodular.* □

The Lovász extension

Lemma: *For each* $w\colon E \to \mathbf{R}_+$*, there uniquely exists a chain*

$$\mathcal{C}\colon A_1 \prec A_2 \prec \cdots \prec A_k$$

of nonempty antichains of P *and* $\lambda_i > 0$ $(i = 1, \cdots, k)$ *such that*

$$w = \lambda_1 \chi_{A_1} + \lambda_2 \chi_{A_2} + \cdots + \lambda_k \chi_{A_k}, \tag{1}$$

where $k \geq 0$. □

- Lovász extension

Let $P = (E, \preceq)$ be a poset and consider an arbitrary function $f\colon \mathcal{A}(P) \to \mathbf{R}$. Define

$$\hat{f}(w) = \sum_{i=1}^{k} \lambda_i f(A_i), \tag{2}$$

where w has the uniquely represented as (1).

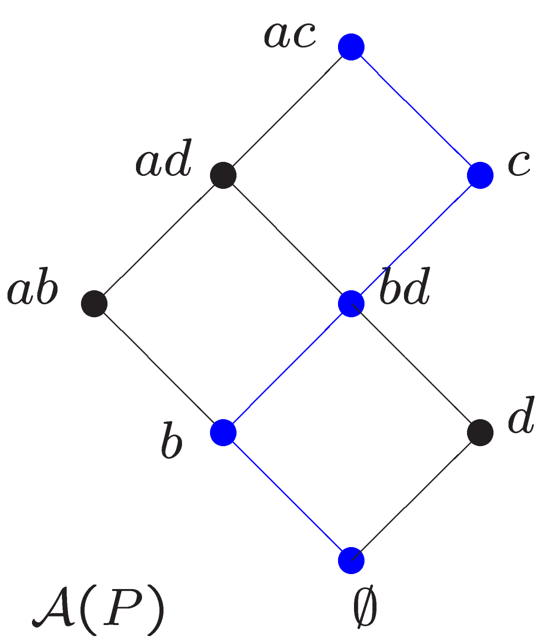
Theorem: *The Lovász extension* $\hat{f}\colon \mathbf{R}_+^E \to \mathbf{R}$ *is convex if and only if* f *is b-submodular.* □

Algorithm for testing a vector for being an extreme point

Theorem (Krüger 2000): *$x \in \mathbf{R}^E$ is an extreme point of* $\mathsf{P}(f)$ *if and only if there exists a chain*

$$\mathcal{C}: \emptyset = A_0 \prec A_1 \prec \cdots \prec A_{n-1} \prec A_n = \max(E)$$

of antichains such that $x(A_i) = f(A_i)$ for $i = 1, \cdots, n$. □



- $A \leftarrow \emptyset$.
- while $\exists B \in \mathcal{A}(P)$ s.t. B covers A and $x(B) = f(B)$ do $A \leftarrow B$.

Lemma: *Let* $A \in \mathcal{A}(P) - \{\max(E)\}$. *Then,* $A \prec\!\cdot\; B$ *if and only if* $B = A \vee e$ *for some* $e \in \min(E - \mathrm{id}(A))$. $\square$

Lemma: *Suppose that* x *is an extreme point of* $\mathrm{P}(f)$. *For each* $A \in \mathcal{A}(P) - \{\max(E)\}$ *with* $x(A) = f(A)$ *there exists an* $e \in \min(E - \mathrm{id}(A))$ *such that* $x(A \vee e) = f(A \vee e)$. $\square$

Theorem: *The following algorithm is valid.* $\square$

Algorithm 2 Extreme

Require: $x \in \mathbf{R}^E$.

Ensure: YES if x is an extreme point of P(f), NO otherwise.

```
A ← ∅.
while ∃e ∈ min(E − id(A)) such that x(A ∨ e) = f(A ∨ e) do
    A ← A ∨ e.
end while
if A = max(E) then
    return YES.
else
    return NO.
end if
```

Conclusion and Remarks

- Characterization by the validity of Dual Greedy Algorithm.

- TDIity.

- Lovász extension.

  - Polynomial time algorithm for b-submodular function minimization.

- Combinatorial Polynomial time algorithm for testing a vector for being a extreme point.

- ◦ Intersection Theorem $\longrightarrow$ holds if P is a rooted forest (Faigle and Kern (2000)).

- Intersection of $\mathrm{P}(f)$ and a box.

- Representation of $\mathrm{P}(f)$ by a submodular flow polyhedron [Fujishige (2000)].