Weak Majorization on Finite Jump Systems†

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Abstract We prove the conjecture posed by A. Tamir [7] that any finite jump system has a least weakly submajorized elements and a least weakly supermajorized element.

Key words Jump system, majorization, separable convex optimization.

1. Definitions and Preliminaries

Majorization and weak majorization
Let us denote the set of reals by $\mathbf{R}$. Let $N = \{1, 2, \cdots, n\}$, where n is a positive integer. For any $x \in \mathbf{R}^N$ let the components $x(i)$ $(i \in N)$ of x are ordered as

$$x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}. \tag{1.1}$$

For $x, y \in \mathbf{R}^N$ if

$$\sum_{i=1}^{j} x_{[i]} \leq \sum_{i=1}^{j} y_{[i]} \quad (j = 1, \cdots, n), \tag{1.2}$$

we say x is *weakly submajorized* by y and denote it by $x \preceq_w y$. If we have $x \preceq_w y$ and $\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$, we say x is *majorized* by y and denote it by $x \preceq y$. For any $x \in \mathbf{R}^N$ let the components $x(i)$ $(i \in N)$ of x are ordered as

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}. \tag{1.3}$$

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For $x, y \in \mathbf{R}^N$ if

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)} \quad (j = 1, \cdots, n), \tag{1.4}$$

we say x is *weakly supermajorized* by y and denote it by $x \preceq^w y$. Note that $x \preceq y$ if and only if $x \preceq_w y$ and $x \preceq^w y$.

The weak submajorization is characterized in terms of convex functions as follows (see [6] for the proof and other characterizations).

Theorem 1.1: *For any $x, y \in \mathbf{R}^N$ x is weakly submajorized (respectively, supermajorized) by y if and only if for any continuous nondecreasing (respectively, nonincreasing) convex function $h\colon \mathbf{R} \to \mathbf{R}$ we have $\sum_{i=1}^n h(x(i)) \leq \sum_{i=1}^n h(y(i))$.* □

For any nonempty subset $S \subseteq \mathbf{R}^N$ an element x in S is called a *least weakly submajorized* (*supermajorized*) *element* of S if $x \preceq_w y$ (respectively, $x \preceq^w y$) holds for each $y \in S$. From Theorem 1.1 we have the following characterization of the least weakly sub- and supermajorization.

Corollary 1.2: *For any nonempty subset S of $\mathbf{R}^N$ x^* is a least weakly submajorized (respectively, supermajorized) element of S if and only if for any continuous nondecreasing (respectively, nonincreasing) convex function $h : \mathbf{R} \to \mathbf{R}$ x^* is an optimal solution for the problem* $\min\{\sum_{i=1}^n h(x(i)) \mid x \in S\}$. □

Jump Systems and Separable Convex Optimization

Define

$$\mathrm{St} = \{\pm\chi_i \mid i \in N\}, \tag{1.5}$$

where $\chi_i : N \to \mathbf{R}$ is defined by $\chi_i(j) = 1$ if $j = i$ and $\chi_i(j) = 0$ otherwise. Each element in St is called a *step*. For $x, y \in \mathbf{R}^N$ a *step from x to y* is a step s such that we have

$$||(x + s) - y||_1 < ||x - y||_1, \tag{1.6}$$

where $||\cdot||_1\colon \mathbf{R}^N \to \mathbf{R}$ is defined by

$$||x||_1 = \sum_{i=1}^{n} |x(i)|. \tag{1.7}$$

The pair $(N, \mathcal{J})$ of N and a nonempty set $\mathcal{J} \subseteq \mathbf{Z}^N$ is called a *jump system* ([3]) on N if the set $\mathcal{J}$ satisfies the following *two step axiom*:

(2SA) For any $x, y \in \mathcal{J}$ and $s \in \mathrm{St}(x, y)$ such that $x + s \notin \mathcal{J}$ there exists a step $t \in \mathrm{St}(x + s, y)$ such that $x + s + t \in \mathcal{J}$.

For a jump system $(N, \mathcal{J})$ on N let us consider the following optimization problem

$$P_g\colon \min\{g(x) \mid x \in \mathcal{J}\}, \tag{1.8}$$

where $g(x) = \sum_{i \in N} g_i(x(i))$ and $g_i\colon \mathrm{R} \to \mathrm{R}$ is a convex function for each $i \in N$. For Problem P_g $x \in \mathcal{J}$ is called a *local optimal solution* if it satisfies the following two condition:

(LO1) For any $s \in \mathrm{St}$ such that $x + s \in \mathcal{J}$ we have $g(x) \leq g(x + s)$.
(LO2) For any $s, t \in \mathrm{St}$ such that $x + s + t \in \mathcal{J}$ we have $g(x) \leq g(x + s + t)$.

The optimality condition for Problem P_g is characterized by the local optimality conditions as the following theorem shows. A jump system $(N, \mathcal{J})$ on N is called *finite* if $\mathcal{J}$ is a finite set.

Theorem 1.3 (Ando, Fujishige and Naitoh [2]): *Suppose that $(N, \mathcal{J})$ is a finite jump system on N. $x \in \mathcal{J}$ is an optimal solution for P_g if and only if x is a local optimal solution.* □

2. The Main Result

Theorem 2.1: *Suppose that $(N, \mathcal{J})$ is a finite jump system on N. $\mathcal{J}$ has a least weakly sub- and supermajorized element.*
(Proof) We prove the existence of a least weakly submajorized element only since the proof of the existence of a least weakly supermajorized element is similar.

We may assume without loss of generality that $\mathcal{J} \subseteq \mathbf{Z}_+^N$, where $\mathbf{Z}_+$ stands for the set of nonnegative integers. Let x^* be an optimal solution for Problem P_f, where the associated objective function $f\colon \mathbf{Z}^N \to \mathbf{R}$ is given by $f(x) = \sum_{i \in N} (x(i))^2$. Then, we must have

$$x^* - \chi_i \notin \mathcal{J} \quad (i \in N) \tag{2.1}$$

and

$$x^* - \chi_i - \chi_j \notin \mathcal{J} \quad (i, j \in N) \tag{2.2}$$

since f is strictly increasing. For any two distinct $i, j \in N$ if $x^* + \chi_i - \chi_j \in \mathcal{J}$ we have

$$\begin{aligned} 0 &\leq f(x^* + \chi_i - \chi_j) - f(x^*) \\ &= (x^*(i) + 1)^2 - x^*(i)^2 + (x^*(j) - 1)^2 - x^*(j)^2 \\ &= 2x^*(i) - 2x^*(j) + 2. \end{aligned} \tag{2.3}$$

It follows that

$$x^*(j) \leq x^*(i) + 1. \tag{2.4}$$

Let $h : \mathbf{R} \to \mathbf{R}$ be given nondecreasing convex function and $g : \mathbf{R}^N \to \mathbf{R}$ be defined as $g(x) = \sum_{i \in N} h(x(i))$ $(x \in \mathbf{R}^N)$. We will show that x^* is also an optimal solution for Problem P_g.

Since h is nondecreasing, we have

$$g(x^*) \le g(x^* + \chi_i) \quad (i \in N) \tag{2.5}$$

and

$$g(x^*) \le g(x^* + \chi_i + \chi_j) \quad (i, j \in N). \tag{2.6}$$

Also, it follows from (2.4) and the convexity of h that for two distinct $i, j \in N$

$$h(x^*(i) + 1) - h(x^*(i)) \ge h(x^*(j)) - h(x^*(j) - 1). \tag{2.7}$$

Therefore, we have

$$g(x^* + \chi_i - \chi_j) \ge g(x^*). \tag{2.8}$$

x^* satisfies (LO1) since we have (2.1) and (2.5) and satisfies (LO2) since we have (2.2), (2.6) and (2.8). That is, x^* is a local optimal solution for Problem P_g and it follows from Theorem 1.3 that is an optimal solution for P_g. However, since h is an arbitrary nondecreasing convex function. It follows from Corollary 1.2 that x^* is a least weakly submajorized element of $\mathcal{J}$. This completes the proof of the present theorem. □

3. Concluding Remarks

We can find a least weakly sub- and supermajorized element in a finite jump system by using an incremental algorithm [2]. The algorithm is designed for solving separable convex programs and is not a polynomial time algorithm. In general, no polynomial time algorithm is known for finding a least weakly sub- and supermajorized elements in a finite jump system. However, when $\mathcal{J}$ is the integral points of a bisubmodular polyhedra [4] (see also [3], [1]), a least weakly sub- and supermajorized elements can be found in polynomial time (see [5]).

References

[1] K. Ando and S. Fujishige: On structures of bisubmodular polyhedra. Report No. 94821, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (April 1994).

[2] K. Ando, S. Fujishige and T. Naitoh: A greedy algorithm for minimizing a separable convex function over a finite jump system. *Journal of the Operations Research Society of Japan* **38** (1995) 362-375.

[3] A. Bouchet and W. H. Cunningham: Delta-matroids, jump systems and bisubmodular polyhedra. *SIAM Journal on Discrete Mathematics* **8** (1995) 17-32.

[4] R. Chandrasekaran and S. N. Kabadi: Pseudomatroids. *Discrete Mathematics* **71** (1988) 205-217.

[5] S. Fujishige: A min-max theorem for bisubmodular polyhedra. Report No. 94818, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (February 1994).

[6] A. W. Marshall and I. Olkin: *Inequalities: Theory of Majorization and its Applications* (Academic Press, New York, 1979).

[7] A. Tamir: Least majorized elements and generalized polymatroids. *Mathematics of Operations Research* **20** (1995) 583-589.