Rockafellar and Wets: Variational Analysis 5.D

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Review of Cosmic Spaces (3.A and 4.F)

We sometimes use the notion of the *extended real* $\mathbb{R}\cup\{+\infty,-\infty\}$. The space $\mathbb{R}^n$ is also *extended* to $\operatorname{csm}\mathbb{R}^n = \mathbb{R}^n \cup \operatorname{hzn}\mathbb{R}^n$, which is called *cosmic closure of* $\mathbb{R}^n$.

For $x \neq \mathbf{0}$ $\operatorname{dir} x$ can be considered as "the point at infinity in the direction of x". The set $\operatorname{hzn}\mathbb{R}^n$ consists of all such points:

$$\operatorname{hzn}\mathbb{R} = \{\operatorname{dir} x \mid \mathbf{0} \neq x \in \mathbb{R}^n\}.$$

Each subset $D \subseteq \operatorname{hzn}\mathbb{R}^n$ can be uniquely represented by a cone $K \subseteq \mathbb{R}^n$ as $D = \operatorname{dir} K$, where

$$\operatorname{dir} K = \{\operatorname{dir} x \mid \mathbf{0} \neq x \in K\}.$$

Hence, each $S \subseteq \operatorname{csm}\mathbb{R}^n$ is written uniquely as $S = C \cup \operatorname{dir} K$, where $C \subseteq \mathbb{R}^m$ is a subset and $K \subseteq \mathbb{R}^m$ is a cone.

Definition (3.1): *For a sequence $\{\tilde{x}^\nu\} \subseteq \operatorname{csm}\mathbb{R}^n$ and a point $\tilde{x} \in \operatorname{csm}\mathbb{R}^n$ we say $\tilde{x}^\nu \to \tilde{x}$ if either*

(i) $\tilde{x} \in \mathbb{R}^n$, $\tilde{x}^\nu \in \mathbb{R}^n$ *for all but finite ν and $\tilde{x}^\nu \to \tilde{x}$ in the ordinary sense.*

(ii) $\tilde{x} = \operatorname{dir} x \in \operatorname{hzn}\mathbb{R}^n$, $\tilde{x}^\nu \in \mathbb{R}^n$ *for all but finite ν and*

$$\exists \lambda^\nu \searrow 0 \colon \lambda^\nu \tilde{x}^\nu \to x. \tag{1}$$

(iii) $\tilde{x} = \operatorname{dir} x \in \operatorname{hzn}\mathbb{R}^n$, $\tilde{x}^\nu = \operatorname{dir} x^\nu \in \operatorname{hzn}\mathbb{R}^n$ *for all but finite ν and*

$$\exists \lambda^\nu > 0 \colon \lambda^\nu x^\nu \to x. \tag{2}$$

(iv) $\tilde{x} = \operatorname{dir} x \in \operatorname{hzn}\mathbb{R}^n$, *both $\{\tilde{x}^\nu\} \cap \mathbb{R}^n$ and $\{\tilde{x}^\nu\} \cap \operatorname{hzn}\mathbb{R}^n$ are infinite, and for $\{\tilde{x}^\nu\} \cap \mathbb{R}^n$* (1) *holds and and for $\{\tilde{x}^\nu\} \cap \operatorname{hzn}\mathbb{R}^n$* (2) *holds.*

□

Definition (in the first paragraph of 4.F): *For $S^\nu \subseteq \operatorname{csm}\mathbb{R}^n$*

$$\limsup_\nu S^\nu = \{x \in \operatorname{csm}\mathbb{R}^n \mid \exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in S^\nu \ (\nu \in N) \colon x^\nu \xrightarrow[N]{} x\}, \tag{3}$$

$$\liminf_\nu S^\nu = \{x \in \operatorname{csm}\mathbb{R}^n \mid \exists N \in \mathcal{N}_\infty, \exists x^\nu \in S^\nu \ (\nu \in N) \colon x^\nu \xrightarrow[N]{} x\}. \tag{4}$$

□

Exercise (4.20'): *We have*

(i) $\limsup_\nu (C^\nu \cup \operatorname{dir} K^\nu) = (\limsup_\nu C^\nu) \cup \operatorname{dir}(\limsup_\nu^\infty C^\nu \cup \limsup_\nu K^\nu)$,

(ii) $\liminf_\nu (C^\nu \cup \operatorname{dir} K^\nu) \supseteq (\liminf_\nu C^\nu) \cup \operatorname{dir}(\liminf_\nu^\infty C^\nu \cup \liminf_\nu K^\nu)$.

If $K^\nu = \{\mathbf{0}\}$ $(\nu \in N)$, we have equality in (ii). □

5.D

Definition (cosmic outer/inner limit): Let $S\colon \mathbb{R}^n \rightrightarrows \operatorname{csm}\mathbb{R}^m$. For $\bar{x} \in \mathbb{R}^n$ the *cosmic outer limit* $\limsup_{x\to\bar{x}} S(x)$ and the *cosmic inner limit* $\liminf_{x\to\bar{x}} S(x)$ at $\bar{x}$ are defined as

$$\limsup_{x\to\bar{x}} S(x) \equiv \bigcup_{x^\nu\to\bar{x}} \limsup_\nu S(x^\nu) \tag{5}$$

$$\liminf_{x\to\bar{x}} S(x) \equiv \bigcap_{x^\nu\to\bar{x}} \liminf_\nu S(x^\nu). \tag{6}$$

□

Definition (horizon outer/inner limit for point-to-set functions): *For $C\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $\bar{x} \in \mathbb{R}^n$ define*

$$\limsup_{x\to\bar{x}} {}^\infty C(x) = \bigcup_{x^\nu\to\bar{x}} \limsup_\nu {}^\infty C(x^\nu), \tag{7}$$

$$\liminf_{x\to\bar{x}} {}^\infty C(x) = \bigcap_{x^\nu\to\bar{x}} \liminf_\nu {}^\infty C(x^\nu). \tag{8}$$

□

Note the difference from $S^\infty(x)$ (see 5(6)).

Let us consider $S\colon \mathbb{R}^n \rightrightarrows \operatorname{csm}\mathbb{R}^m$, where $\operatorname{csm}\mathbb{R}^m = \mathbb{R}^m \cup \operatorname{dir}\mathbb{R}^m$ so that for each x $S(x)$ is written as $S(x) = C(x) \cup \operatorname{dir} K(x)$, where $C(x) \subseteq \mathbb{R}^m$ is a subset and $K(x) \subseteq \mathbb{R}^m$ is a cone.

Proposition: *For $S\colon \mathbb{R}^n \rightrightarrows \operatorname{csm}\mathbb{R}^m$ and $\bar{x} \in \mathbb{R}^n$ we have*

$$\limsup_{x\to\bar{x}} S(x) = \limsup_{x\to\bar{x}} C(x) \bigcup \operatorname{dir} \left(\limsup_{x\to\bar{x}} {}^\infty C(x) \cup \limsup_{x\to\bar{x}} K(x) \right), \tag{9}$$

$$\liminf_{x\to\bar{x}} S(x) \supseteq \liminf_{x\to\bar{x}} C(x) \bigcup \operatorname{dir} \left(\liminf_{x\to\bar{x}} {}^\infty C(x) \cup \liminf_{x\to\bar{x}} K(x) \right). \tag{10}$$

For $S\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ equality holds in (10).

(Proof)

$$\limsup_{x\to\bar{x}} S(x) \equiv \bigcup_{x^\nu\to\bar{x}} \limsup_\nu S(x^\nu) \tag{11}$$

$$= \bigcup_{x^\nu\to\bar{x}} (\limsup_\nu C(x^\nu) \bigcup \operatorname{dir} (\limsup_\nu^\infty C(x^\nu) \cup \limsup_\nu K(x^\nu))) \tag{12}$$

$$= \bigcup_{x^\nu\to\bar{x}} \limsup_\nu C(x^\nu) \bigcup \operatorname{dir} (\bigcup_{x^\nu\to\bar{x}} \limsup_\nu^\infty C(x^\nu) \bigcup \bigcup_{x^\nu\to\bar{x}} \limsup_\nu K(x^\nu)) \tag{13}$$

$$= \limsup_{x\to\bar{x}} C(x) \bigcup \operatorname{dir} \left(\limsup_{x\to\bar{x}} {}^\infty C(x) \cup \limsup_{x\to\bar{x}} K(x) \right) \tag{14}$$

and

$$\begin{aligned}
\liminf_{x\to\bar{x}} S(x) &\equiv \bigcap_{x^\nu\to\bar{x}} \liminf_\nu S(x^\nu) && (15)\\
&\supseteq \bigcap_{x^\nu\to\bar{x}} \left(\liminf_\nu C(x^\nu) \bigcup \operatorname{dir}\left(\liminf_\nu^\infty C(x^\nu) \bigcup \liminf_\nu K(x^\nu)\right)\right) && (16)\\
&= \bigcap_{x^\nu\to\bar{x}} \liminf_\nu C(x^\nu) \\
&\quad \bigcup \operatorname{dir}\left(\bigcap_{x^\nu\to\bar{x}} \left(\liminf_\nu^\infty C(x^\nu) \bigcup \liminf_\nu K(x^\nu)\right)\right) && (17)\\
&\supseteq \bigcap_{x^\nu\to\bar{x}} \liminf_\nu C(x^\nu) \\
&\quad \bigcup \operatorname{dir}\left(\bigcap_{x^\nu\to\bar{x}} \liminf_\nu^\infty C(x^\nu) \bigcup \bigcap_{x^\nu\to\bar{x}} \liminf_\nu K(x^\nu)\right) && (18)\\
&= \liminf_{x\to\bar{x}} C(x) \\
&\quad \bigcup \operatorname{dir}\left(\liminf_{x\to\bar{x}} {}^\infty C(x) \cup \liminf_{x\to\bar{x}} K(x)\right). && (19)
\end{aligned}$$

□

Definition (cosmic continuity/semicontinuity): For $S\colon \mathbb{R}^n \rightrightarrows \operatorname{csm} \mathbb{R}^m$, S is *cosmically continuous* at $\bar{x}$ if

$$\limsup_{x\to\bar{x}} S(x) \subseteq S(\bar{x}) \subseteq \liminf_{x\to\bar{x}} S(x), \tag{20}$$

or equivalently, if

$$\limsup_\nu S(x^\nu) \subseteq S(\bar{x}) \subseteq \liminf_\nu S(x^\nu) \tag{21}$$

whenever $x^\nu \to \bar{x}$.

S is *cosmically osc at* $\bar{x}$ if

$$\limsup_{x\to\bar{x}} S(x) \subseteq S(\bar{x}) \tag{22}$$

or equivalently, if

$$\limsup_\nu S(x^\nu) \subseteq S(\bar{x}) \tag{23}$$

for $x^\nu \to \bar{x}$.

S is *cosmically isc at* $\bar{x}$ if

$$S(\bar{x}) \subseteq \liminf_{x\to\bar{x}} S(x), \tag{24}$$

or equivalently, if

$$S(\bar{x}) \subseteq \liminf_\nu S(x^\nu). \tag{25}$$

for each $x^\nu \to \bar{x}$. □

Proposition (5.27'): *$S\colon \mathbb{R}^m \rightrightarrows \operatorname{csm} \mathbb{R}^m$ is cosmically osc at $\bar{x}$ if and only if*

$$\limsup_{x\to\bar{x}} C(x) \subseteq C(\bar{x}),\ \limsup_{x\to\bar{x}} {}^\infty C(x) \bigcup \limsup_{x\to\bar{x}} K(x) \subseteq K(\bar{x}).$$

□

Counterexample (for 5.27 Proposition and 4.20 Exercise): *Let us consider* $S\colon \mathbb{R} \rightrightarrows \operatorname{csm}\mathbb{R} = \mathbb{R}\cup\{\infty,-\infty\}$ *defined as*

$$S(x) = \begin{cases} \{x^{-1}\}\cup\{\infty\} & \text{if } x<0,\\ \emptyset\cup\{\infty,-\infty\} & \text{if } x=0,\\ \{x^{-1}\}\cup\{-\infty\} & \text{if } x>0. \end{cases} \tag{26}$$

S is cosmically isc (actually cosmically continuous) at 0 *but*

$$\liminf_{x\to 0}{}^{\infty} C(x) = \{0\},\ \liminf_{x\to 0} K(x) = \{0\}.$$

(Proof) Let us consider an arbitrary sequence $\{x^\nu\}$ converging to 0. Define a sequence $\{u^\nu\}$ by

$$u^\nu = \begin{cases} +\infty & \text{if } x^\nu < 0,\\ \frac{1}{x^\nu} & \text{if } x^\nu > 0,\\ +\infty & \text{otherwise.} \end{cases} \tag{27}$$

Then, $u^\nu \in S(x^\nu)$ for every ν and $u^\nu \to +\infty$. Therefore, $+\infty \in \liminf_\nu S(x^\nu)$. Similarly, $-\infty \in \liminf_\nu S(x^\nu)$. This shows $\{+\infty,-\infty\}\subseteq \liminf_{x\to 0} S(x)$, and hence, S is cosmically isc.

On the other hand, let us consider a sequence $\{x^\nu\}$ defined as $x^\nu = \frac{(-1)^\nu}{\nu}$. For this sequence,

$$\liminf{}^{\infty}_{\nu} C(x^\nu) = \{0\},\ \liminf_\nu K(x^\nu) = \{0\}.$$

This sequence is also a counterexample for 4.20. (More explicitly, the sequence is given by

$$S^\nu = \begin{cases} \{\nu\}\cup\{-\infty\} & \text{if } \nu \text{ is odd}\\ \{-\nu\}\cup\{\infty\} & \text{if } \nu \text{ is even.} \end{cases} \tag{28}$$

) □

Definition (total continuity/semicontinuity): $S\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *totally continuous* at $\bar{x}$ if $S(\bar{x})$ is closed and

$$S(x^\nu) \xrightarrow{\text{t}} S(\bar{x}) \text{ whenever } x^\nu \to \bar{x},$$

that is,

$$\operatorname{csm} S(x^\nu) \xrightarrow{\text{c}} \operatorname{csm} S(x) \text{ whenever } x^\nu \to \bar{x},$$

which is equivalent to

$$\lim_\nu S(x^\nu) = S(\bar{x}),\ \limsup{}^{\infty}_{\nu} S(x^\nu) \subseteq S(\bar{x})^\infty \text{ whenever } x^\nu \to \bar{x}$$

by Proposition 4.24.

S is *totally osc* if

$$\limsup_{x\to\bar{x}} S(x) \subseteq S(\bar{x}),\ \limsup_{x\to\bar{x}}{}^{\infty} S(x) \subseteq S(\bar{x})^\infty,$$

i.e., for every sequence $x^\nu \to \bar{x}$

$$\limsup_\nu S(x^\nu) \subseteq S(\bar{x}),\ \limsup{}^{\infty}_{\nu} S(x^\nu) \subseteq S(\bar{x})^\infty.$$

S is *totally isc* if

$$S(\bar{x}) \subseteq \liminf_{x\to\bar{x}} S(x),\ S(\bar{x})^\infty \subseteq \liminf_{x\to\bar{x}}{}^{\infty} S(x),$$

that is, for each sequence $x^\nu \to \bar{x}$ we have

$$S(\bar{x}) \subseteq \liminf_\nu S(x^\nu),\ S(\bar{x})^\infty \subseteq \liminf{}^{\infty}_{\nu} S(x^\nu).$$

□

However,

Remark: *Total inner semicontinuity is equivalent to ordinary inner semicontinuity.*

(Proof) See Exercise 4.21(c) (but not 4.20). □

Proposition (5.29): *$S:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is totally continuous if and only if it is continuous and*

$$\limsup_{x\to\bar{x}} {}^{\infty} S(x) \subseteq S(\bar{x})^{\infty}. \tag{29}$$

Total continuity at $\bar{x}$ is automatic from continuity at $\bar{x}$ when

(i) *S is convex-valued on a neighborhood of $\bar{x}$ and $S(\bar{x}) \neq \emptyset$, or*

(ii) *S is cone-valued on a neighborhood of $\bar{x}$, or*

(iii) *S is locally bounded at $\bar{x}$.*

(Proof) We will show (i) only since the others can be shown similarly. Suppose that for some neighborhood V of $\bar{x}$ S is convex-valued, that S is continuous at $\bar{x}$ and $S(\bar{x}) \neq \emptyset$.

For each sequence $\{x^\nu\}$ converging to $\bar{x}$ there exists a number ν_0 such that $x^\nu \in V$ for $\nu \geq \nu_0$, and hence, $S(x^\nu)$ is convex for $\nu \geq \nu_0$. It follows from Theorem 4.25 that $S(x^\nu) \xrightarrow{\text{t}} S(\bar{x})$. □

Exercise (5.30): *Consider a mapping $S:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and subsets $C^\nu \subseteq \mathbb{R}^n$.*

(a) *If S is isc, one has $\liminf_\nu S(C^\nu) \supseteq S(\liminf_\nu C^\nu)$.*

(b) *If S is osc, $\limsup_\nu S(C^\nu) \subseteq S(\limsup_\nu C^\nu)$ provided that S^{-1} is locally bounded, or alternatively that $(S^\infty)^{-1}(\mathbf{0}) \cap \limsup_\nu^\infty C^\nu = \{\mathbf{0}\}$.*

(c) *If S is continuous, one has $S(C^\nu) \to S(C)$ whenever $C^\nu \to C$ and S^{-1} is locally bounded, or alternatively, whenever $C^\nu \xrightarrow{\text{t}} C$ and $(S^\infty)^{-1}(\mathbf{0}) \cap C^\infty = \{\mathbf{0}\}$.*

(d) *If S is totally continuous, one has $S(C^\nu) \xrightarrow{\text{t}} S(C)$ whenever $C^\nu \xrightarrow{\text{t}} C$, $(S^\infty)^{-1} \cap C^\infty = \{\mathbf{0}\}$ and $(S^\infty)(C^\infty) \subseteq S(C)^\infty$.*

(Proof) (a) Supposing $\bar{u} \in S(\liminf_\nu C^\nu)$, we have

$$\begin{aligned} &\bar{u} \in S(\liminf_\nu C^\nu) \\ &\quad \Rightarrow \quad \exists \bar{x} \in \liminf_\nu C^\nu : \bar{u} \in S(\bar{x}) \\ &\quad \Rightarrow \quad \exists N \in \mathcal{N}_\infty, \exists x^\nu \in C^\nu \ (\nu \in N) : x^\nu \xrightarrow[N]{} \bar{x}, \\ &\quad \quad \bar{u} \in S(\bar{x}). \end{aligned}$$

Since S is isc, we have

$$\bar{u} \in S(\bar{x}) \subseteq \liminf_\nu S(x^\nu).$$

However, since $x^\nu \in C^\nu$, we have $\liminf_\nu S(x^\nu) \subseteq \liminf_\nu S(C^\nu)$.

(b) Let $\bar{u} \in \limsup_\nu S(C^\nu)$. Then,

$$\exists N \in \mathcal{N}_\infty^\#, \exists u^\nu \in S(C^\nu) \ (\nu \in N) : u^\nu \xrightarrow[N]{} \bar{u}$$

or

$$\exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in C^\nu \ (\nu \in N) : u^\nu \in S(x^\nu), u^\nu \xrightarrow[N]{} \bar{u}.$$

[**Case I:** S^{-1} is locally bounded.] If S^{-1} is locally bounded, then there exists a neighborhood V of $\bar{u}$ such that $S^{-1}(V)$ is bounded. Since we have for some $N' \in \mathcal{N}_\infty^\#$ with $N' \subseteq N$ that $u^\nu \in V$ $(\nu \in N')$, we must have $S^{-1}(u^\nu) \subseteq S^{-1}(V)$ $(\nu \in N')$, the sequence $\{x^\nu\}_{N'}$ is bounded.

[**Case II**: $(S^\infty)^{-1}(0) \cap \limsup_\nu^\infty C^\nu = \{\mathbf{0}\}$. (cf. the argument in the proof of Theorem 4.26)] We will show that $\{x^\nu\}_N$ is bounded. Suppose, on the contrary that $\{x^\nu\}_N$ is unbounded. Then, there exists a subsequence $\{x^\nu\}_{N'}$ of $\{x^\nu\}_N$, $\lambda^\nu \searrow 0$ and $x \neq \mathbf{0}$ such that $\lambda^\nu x^\nu \underset{N'}{\longrightarrow} x$. Then, we have $x \in \limsup_\nu^\infty C^\nu$ by definition. We also have $\lambda^\nu(x^\nu, u^\nu) \underset{N'}{\longrightarrow} (x, \mathbf{0})$. Since $x^\nu \in S^{-1}(u^\nu)$ for $\nu \in N'$, we have $x \in (S^{-1})^\infty(\mathbf{0})$, and hence, $x = \mathbf{0}$, a contradiction.

Therefore, in any case there exists a converging subsequence $\{x^\nu\}_{N''}$ of $\{x^\nu\}_{N'}$. Let $x^\nu \underset{N''}{\longrightarrow} \bar{x}$. Then, $\bar{x} \in \limsup_\nu C^\nu$.

By renumbering, we assume $\{x^\nu\}_{N''}$ is indexed by the natural number. Then, $x^\nu \to \bar{x}, u^\nu \to \bar{u}$ and $u^\nu \in S(x^\nu)$. Therefore, $\bar{u} \in \limsup_{x \to \bar{x}} S(x)$. On the other hand, since S is osc,

$$\bar{u} \in \limsup_{x \to \bar{x}} S(x) \subseteq S(\bar{x}) \subseteq S(\limsup_\nu C^\nu).$$

(c) We have from (a) and (b) that

$$\limsup_\nu S(C^\nu) \subseteq S(\limsup_\nu C^\nu) \subseteq S(\liminf_\nu C^\nu) \subseteq \liminf_\nu S(C^\nu).$$

(d) It follows from (c) that $S(C^\nu) \to S(C)$. We must show that

$$\limsup_\nu^\infty S(C^\nu) \subseteq S^\infty(C^\infty) \ (\subseteq S(C)^\infty)$$

(see Proposition 4.24). Suppose that $\bar{u} \in \limsup_\nu^\infty S(C^\nu)$. Then, we have

$$\exists N \in \mathcal{N}_\infty^\#, \exists u^\nu \in S(C^\nu) \ (\nu \in N), \exists \lambda^\nu \searrow 0 \colon \lambda^\nu u^\nu \underset{N}{\longrightarrow} \bar{u},$$

or

$$\exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in C^\nu \ (\nu \in N), \exists \lambda^\nu \searrow 0 \colon u^\nu \in S(x^\nu), \lambda^\nu u^\nu \underset{N}{\longrightarrow} \bar{u}.$$

If $\bar{u} = \mathbf{0}$, then we have $\bar{u} = \mathbf{0} \in S^\infty(\mathbf{0}) \subseteq S^\infty(C^\infty)$, and we are done. Hence, we assume $\bar{u} \neq \mathbf{0}$.

[**Case I**: $\{x^\nu\}_N$ is bounded] We have

$$\lambda^\nu x^\nu \to 0 \in C^\infty,$$

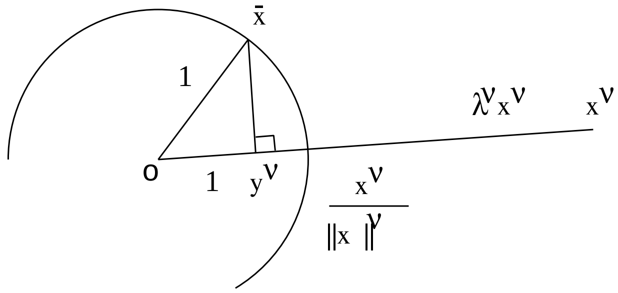
and hence, $\bar{u} \in S^\infty(C^\infty)$.

[**Case II**: $\{x^\nu\}_N$ is unbounded] Since $\{(x^\nu, u^\nu)\}_N$ is unbounded, there exists a subsequence $\{(x^\nu, u^\nu)\}_{N'}$ of $\{(x^\nu, u^\nu)\}_N$, $\mu^\nu \searrow 0$ and $(\bar{x}, \bar{u}') \neq \mathbf{0}$ such that $\mu^\nu(x^\nu, u^\nu) \underset{N}{\longrightarrow} (\bar{x}, \bar{u}')$. If $\bar{u}' = \mathbf{0}$, then we have $\mathbf{0} \neq \bar{x} \in (S^{-1})^\infty(\mathbf{0}) \cap C^\infty$, a contradiction. Therefore, we have $\bar{u}' \neq \mathbf{0}$. It follows from the lemma below that $\bar{u}' = \gamma\bar{u}$ for some $\gamma > 0$. Hence, we can assume, by scaling μ^ν if necessary, that $\bar{u}' = \bar{u}$. Then, we have $\bar{u} \in S^\infty(\bar{x})$ and $\bar{x} \in C^\infty$, and hence, $\bar{u} \in S^\infty(C^\infty)$. □

Lemma: *Let us consider a sequence $\{x^\nu\}$. Then, there exists $\lambda^\nu \searrow 0$ and $\bar{x}$ with $\|\bar{x}\| = 1$ such that $\lambda^\nu x^\nu \to \bar{x}$ if and only if $\frac{x^\nu}{\|x^\nu\|} \to \bar{x}$.*

(Proof) It suffices to prove the "only if" part.

For each ν let y^ν be the projection of $\bar{x}$ onto the line through x^ν and the origin. Then, we have $\|\bar{x} - y^\nu\| \leq \|\bar{x} - \lambda^\nu x^\nu\|$, and hence, $y^\nu \to \bar{x}$ $(\nu \to \infty)$.



Since $\frac{x^\nu}{\|x^\nu\|} = \frac{y^\nu}{\|y^\nu\|}$ for each ν, it follows that $\frac{x^\nu}{\|x^\nu\|} \to \bar{x}$ $(\nu \to \infty)$. □

Appendix

(Proof of Exercise 4.20') (i) Suppose $\bar{x} \in \limsup_\nu (C^\nu \cup \operatorname{dir} K^\nu)$. Then,

$$\exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in C^\nu \cup \operatorname{dir} K^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} \bar{x}.$$

[**Case I:** $\bar{x} \in \mathbb{R}^n$.] In this case, for a sufficiently large ν_0 we have $\nu \geq \nu_0, \nu \in N$ implies $x^\nu \in C^\nu$. Hence, $\bar{x} \in \limsup_\nu C^\nu$.

[**Case II:** $\bar{x} \in \operatorname{hzn} \mathbb{R}^n$.] Suppose $\bar{x} = \operatorname{dir} x$ for some $x \neq \mathbf{0}$.

[**Case II-a:** There exist infinitely many $\nu \in N$ such that $x^\nu \in C^\nu$.] Then, by definition, there exists $\lambda^\nu \searrow 0$ such that $\lambda^\nu x^\nu \underset{N}{\longrightarrow} x$. Hence, $x \in \limsup_\nu^\infty C^\nu$. Therefore, $\bar{x} = \operatorname{dir} x \in \operatorname{dir} \limsup_\nu^\infty C^\nu$.

[**Case II-b:** There exists infinitely many $\nu \in N$ such that $x^\nu \in \operatorname{dir} K^\nu$.] In this case, for each such ν there exists $y^\nu \in K^\nu$ such that $x^\nu = \operatorname{dir} y^\nu$. Also, we have $\lambda^\nu y^\nu \to x$ for some $\lambda^\nu > 0$ by definition of convergence of direction points. Therefore, $x \in \limsup_\nu K^\nu$, and hence, $\bar{x} \in \operatorname{dir} \limsup_\nu K^\nu$.

We thus have shown that $\limsup_\nu (C^\nu \cup \operatorname{dir} K^\nu) \subseteq (\limsup_\nu C^\nu) \cup \operatorname{dir} (\limsup_\nu^\infty C^\nu \cup \limsup_\nu K^\nu)$.

Conversely, suppose that $\bar{x} \in (\limsup_\nu C^\nu) \cup \operatorname{dir} (\limsup_\nu^\infty C^\nu \cup \limsup_\nu K^\nu)$. If $\bar{x} \in (\limsup_\nu C^\nu)$, we apparently have $\bar{x} \in \limsup_\nu (C^\nu \cup \operatorname{dir} K^\nu)$.

If $\bar{x} \in \operatorname{dir} (\limsup_\nu^\infty C^\nu \cup \limsup_\nu K^\nu)$, we have $\bar{x} = \operatorname{dir} x$ for some $x \in \limsup_\nu^\infty C^\nu \cup \limsup_\nu K^\nu$ with $x \neq \mathbf{0}$. In case of $x \in \limsup_\nu K^\nu$, we have

$$\exists N \in \mathcal{N}_\infty^\#, \exists y^\nu \in K^\nu \ (\nu \in N) \colon y^\nu \underset{N}{\longrightarrow} x,$$

and hence,

$$\exists \operatorname{dir} y^\nu \in \operatorname{dir} K^\nu \ (\nu \in N) \colon \operatorname{dir} y^\nu \underset{N}{\longrightarrow} \operatorname{dir} x.$$

Therefore, $\bar{x} = \operatorname{dir} x \in \limsup_\nu \operatorname{dir} K^\nu$.

In the other case, we have

$$\exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in C^\nu \ (\nu \in N), \exists \lambda^\nu \searrow 0 \colon \lambda^\nu x^\nu \underset{N}{\longrightarrow} x.$$

By definition, we have

$$\exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in C^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} \operatorname{dir} x,$$

and hence, we have $\bar{x} = \operatorname{dir} x \in \limsup_\nu C^\nu$.

(ii) Suppose $\bar{x} \in (\liminf_\nu C^\nu) \cup \operatorname{dir} (\liminf_\nu^\infty C^\nu \cup \liminf_\nu K^\nu)$.

If $\bar{x} \in \liminf_\nu C^\nu$, then it is clear that $x \in \liminf_\nu (C^\nu \cup \operatorname{dir} K^\nu)$. Suppose $\bar{x} \in \operatorname{dir} (\liminf_\nu^\infty C^\nu \cup \liminf_\nu K^\nu)$. Then, there exists $x \in \liminf_\nu^\infty C^\nu \cup \liminf_\nu K^\nu$ such that $\bar{x} = \operatorname{dir} x$.

[Case: $x \in \liminf_\nu^\infty C^\nu$.] We have

$$\exists N \in \mathcal{N}_\infty, \exists x^\nu \in C^\nu \ (\nu \in N), \exists \lambda^\nu \searrow 0 \colon \lambda^\nu x^\nu \xrightarrow[N]{} x,$$

and hence,

$$\exists x^\nu \in C^\nu \ (\nu \in N) \colon x^\nu \xrightarrow[N]{} \operatorname{dir} x.$$

Therefore, $\bar{x} = \operatorname{dir} x \in \liminf_\nu C^\nu$.
[Case: $x \in \liminf_\nu K^\nu$.] We have

$$\exists N \in \mathcal{N}, \exists x^\nu \in K^\nu \ (\nu \in N) \colon x^\nu \xrightarrow[N]{} x,$$

that is,

$$\exists \operatorname{dir} x^\nu \in \operatorname{dir} K^\nu \ (\nu \in N) \colon \operatorname{dir} x^\nu \xrightarrow[N]{} \operatorname{dir} x.$$

Hence, $\bar{x} = \operatorname{dir} x \in \liminf_\nu \operatorname{dir} K^\nu$.

(iii) Suppose $K^\nu = \{\mathbf{0}\}$ for each ν and let $\bar{x} \in \liminf_\nu C^\nu$. If $\bar{x} \in \mathbb{R}^n$, then

$$\exists N \in \mathcal{N}_\infty, \exists x^\nu \in C^\nu \ (\nu \in N) \colon x^\nu \xrightarrow[N]{} \bar{x},$$

and hence, $\bar{x} \in \liminf_\nu C^\nu$. If $\bar{x} \in \operatorname{hzn} \mathbb{R}^n$, then for some $x \neq \mathbf{0}$ we have $\bar{x} = \operatorname{dir} x$ and

$$\exists N \in \mathcal{N}_\infty, \exists x^\nu \in C^\nu \ (\nu \in N) \colon x^\nu \xrightarrow[N]{} \operatorname{dir} x,$$

and hence,

$$\exists x^\nu \in C^\nu \ (\nu \in N), \exists \lambda^\nu \searrow 0 \colon \lambda^\nu x^\nu \xrightarrow[N]{} x.$$

Therefore, $x \in \liminf_\nu^\infty C^\nu$. Then, we have $\bar{x} = \operatorname{dir} x \in \operatorname{dir} \ \liminf_\nu^\infty C^\nu$. $\square$