Rockafellar and Wets: Variational Analysis

Kazutoshi Ando

2000.10.4

3.B: Horizon Cones

Exercise (3.4): *$C \cup \mathrm{dir}\, K \subseteq \mathrm{csm}\, \mathbb{R}^n$ is closed if and only if C and K are closed in $\mathbb{R}^n$ and $C^\infty \subseteq K$. In general, its cosmic closure is*

$$\mathrm{csm}\,(C \cup \mathrm{dir}\, K) = \mathrm{cl}\, C \cup \mathrm{dir}\,(C^\infty \cup \mathrm{cl}\, K).$$

(Proof) ($\subseteq$) Let us consider a converging sequence $\{\tilde{x}^\nu\} \subseteq C \cup \mathrm{dir}\, K$ with $\tilde{x}^\nu \to \bar{x} \in \mathrm{csm}\, \mathbb{R}^n$. If $\bar{x} \in \mathbb{R}^n$, then for all but finite ν we have $\bar{x}^\nu \in C$, and hence, we have $\bar{x} \in \mathrm{cl}\, C$. If $\bar{x} \in \mathrm{hzn}\, \mathbb{R}^n$, then for some $x \neq \mathbf{0}$ we have $\bar{x} = \mathrm{dir}\, x$. Define

$$N_1 = \{\nu \mid \tilde{x}^\nu \in C\}$$

and

$$N_0 = \{\nu \mid \nu \in N, \tilde{x}^\nu \in \mathrm{dir}\, K\}.$$

[Case: N_1 is infinite] Then, by definition, we have

$$\exists \lambda^\nu \searrow 0 : \lambda^\nu x^\nu \to x,$$

and hence, $x \in C^\infty$. Therefore, $\bar{x} = \mathrm{dir}\, x \in \mathrm{dir}\, C^\infty$.

[Case: N_0 is infinite] Then, by difinition, for each $\nu \in N_0$ there exists $x^\nu \in K$ such that $\bar{x}^\nu = \mathrm{dir}\, x^\nu$ and we have

$$\exists \lambda^\nu > 0 : \lambda^\nu x^\nu \to x.$$

Therefore, $x \in \mathrm{cl}\, K$, and hence, $\bar{x} = \mathrm{dir}\, x \in \mathrm{dir}\, \mathrm{cl}\, K$.

($\supseteq$) Suppose $\bar{x} \in \mathrm{cl}\, C \cup \mathrm{dir}\,(C^\infty \cup clK)$. If $\bar{x} \in \mathrm{cl}\, C$, then $\bar{x}$ is the limit point of a sequence in C. Therefore, $x \in \mathrm{csm}\,(C \cup \mathrm{dir}\, K)$.

Suppose $\bar{x} \in \mathrm{dir}\,(C^\infty \cup \mathrm{cl}\, K)$. Then, there exists $x \neq \mathbf{0}$ such that $\bar{x} = \mathrm{dir}\, x$.
[Case: $x \in C^\infty$] We have

$$\exists \lambda^\nu \searrow 0 \colon \lambda^\nu x^\nu \to x,$$

and hence, $x^\nu \to \mathrm{dir}\, x = \bar{x}$. Therefore, $\bar{x} \in \mathrm{csm}\, C$.
[Case: $x \in \mathrm{cl}\, K$] Then, there exists a sequence $\{x^\nu\}$ in K converging to x. This means

$$\mathrm{dir}\, x^\nu \in \mathrm{dir}\, K \ (\nu \in \mathbb{N}), \mathrm{dir}\, x^\nu \to \mathrm{dir}\, x,$$

and hence, $\bar{x} = \mathrm{dir}\, x \in \mathrm{csm}\, \mathrm{dir}\, K$. $\square$

Proposition: *For a general subset of $\mathrm{csm}\, \mathbb{R}^n$, written as $C \cup \mathrm{dir}\, K$ for a set $C \subseteq \mathbb{R}^n$ and a cone $K \subseteq \mathbb{R}^n$, let*

$$\mathcal{G}(C, K) = \left\{ \lambda \begin{bmatrix} x \\ -1 \end{bmatrix} \mid x \in C, \lambda > 0 \right\} \cup \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in K \right\}.$$

$C \cup \mathrm{dir}\, K$ is cosmically closed if and only if $\mathcal{G}(C, K)$ is closed.

(Proof) ("if" part:) Suppose that $\mathcal{G}(C, K)$ is closed. Then, it is obvious (?) that C and K are closed. It suffices to show that $C^\infty \subseteq K$.

Let $x \in C^\infty$. We have

$$\exists x^\nu \in C, \exists \lambda^\nu \searrow 0 \colon \lambda^\nu x^\nu \to x.$$

Therefore, we have

$$\lambda^\nu \begin{bmatrix} x^\nu \\ -1 \end{bmatrix} \in \mathcal{G}(C,K), \lambda^\nu \begin{bmatrix} x^\nu \\ -1 \end{bmatrix} \to \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

Since $\mathcal{G}(C,K)$is closed, we must have $\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{G}(C,K)$, and hence, $x \in K$.

("only if" part:) Suppose that csm $(C \cup \mathrm{dir}\, K)$ is closed. Let us consider a sequence

$$\begin{bmatrix} x^\nu \\ -\gamma^\nu \end{bmatrix} \in \mathcal{G}(C,K) \text{ with } \begin{bmatrix} x^\nu \\ -\gamma^\nu \end{bmatrix} \to \begin{bmatrix} x \\ -\gamma \end{bmatrix}.$$

[Case: $\gamma > 0$] Then, we have for all but finite ν that $\gamma^\nu > 0$. Therefore,

$$\frac{x^\nu}{\gamma^\nu} \in C, \frac{x^\nu}{\gamma^\nu} \to \frac{x}{\gamma}.$$

Since C is closed, we have $\frac{x}{\gamma} \in C$. It follows that $\begin{bmatrix} x \\ -\gamma \end{bmatrix} \in \mathcal{G}(C,K)$.

[Case: $\gamma = 0$] If for infinitely many ν we have $\gamma^\nu > 0$, we have

$$\frac{x^\nu}{\gamma^\nu} \in C,\ \gamma^\nu \searrow 0,\ \gamma^\nu \frac{x^\nu}{\gamma^\nu} \to x.$$

Therefore, $x \in C^\infty$. Then, we have $x \in K$ since csm $(C \cup \mathrm{dir}\, K)$ is closed. Hence, we have $\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{G}(C,K)$.

If for infinitely many ν we have $\gamma^\nu = 0$, then we have

$$x^\nu \in K, x^\nu \to x.$$

It follows from the closedness of K that $x \in K$, and hence, $\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{G}(C,K)$. □

3.F: Cosmic Convexity

Exercise (3.44): *For a general subset of* csm $\mathbb{R}^n$*, written as* $C \cup \mathrm{dir}\, K$ *for a set* $C \subseteq \mathbb{R}^n$ *and a cone* $K \subseteq \mathbb{R}^n$*, one has*

$$\mathrm{con}\,(C \cup \mathrm{dir}\, K) = (\mathrm{con}\, C + \mathrm{con}\, K) \cup \mathrm{dir}\,(\mathrm{con}\, K).$$

(Proof) For $C \cup \mathrm{dir}\, K \subseteq \mathrm{csm}\, \mathbb{R}^n$ define

$$\mathcal{G}(C,K) = \left\{ \lambda \begin{bmatrix} x \\ -1 \end{bmatrix} \mid x \in C, \lambda > 0 \right\} \cup \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in K \right\}.$$

If we can show that

$$\mathcal{G}(\mathrm{con}\, C + \mathrm{con}\, K, \mathrm{con}\, K) = \mathrm{con}\, \mathcal{G}(C,K),$$

it will follow that for any convex subset $C' \cup \mathrm{dir}\, K'$ such that $C \subseteq C'$ and $K \subseteq K'$, we have $\mathcal{G}(C,K) \subseteq \mathcal{G}(C',K')$ since $\mathcal{G}(C',K')$ is concex due to 3.42. Therefore, we have

$$\mathcal{G}(\mathrm{con}\, C + \mathrm{con}\, K, \mathrm{con}\, K) = \mathrm{con}\, \mathcal{G}(C,K) \subseteq \mathcal{G}(\mathcal{C}', \mathcal{K}'),$$

and hence, $\mathrm{con}\, C + \mathrm{con}\, K \subseteq C', \mathrm{con}\, K \subseteq K'$.

Now, let us show that

$$\mathcal{G}(\mathrm{con}\, C + \mathrm{con}\, K, \mathrm{con}\, K) = \mathrm{con}\, \mathcal{G}(C,K).$$

Let

$$\begin{bmatrix} x \\ -\xi \end{bmatrix} \in \mathrm{con}\, \mathcal{G}(C,K).$$

Then, we have

$$\exists \begin{bmatrix} x_1 \\ -\xi_1 \end{bmatrix}, \cdots, \begin{bmatrix} x_k \\ -\xi_k \end{bmatrix}, \begin{bmatrix} y_1 \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} y_l \\ 0 \end{bmatrix} \in \mathcal{G}(C,K) \text{ with } \xi_i > 0\ (i = 1, \cdots, k),$$

and

$$\exists\lambda_1,\cdots,\lambda_k,\mu_1,\cdots,\mu_l \geq 0 \text{ with } \sum_{i=1}^{k}\lambda_i + \sum_{j=1}^{l}\mu_j = 1$$

such that

$$\begin{bmatrix} x \\ -\xi \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ -\xi_1 \end{bmatrix} + \cdots + \lambda_k \begin{bmatrix} x_k \\ -\xi_k \end{bmatrix} + \mu_1 \begin{bmatrix} y_1 \\ 0 \end{bmatrix} + \cdots + \mu_l \begin{bmatrix} y_l \\ 0 \end{bmatrix}$$

[**Case 1:**] $k = 0$ (or $\xi = 0$).

$$\begin{bmatrix} x \\ -\xi \end{bmatrix} = \mu_1 \begin{bmatrix} y_1 \\ 0 \end{bmatrix} + \cdots + \mu_l \begin{bmatrix} y_l \\ 0 \end{bmatrix} = \begin{bmatrix} \mu_1 y_l + \cdots + \mu_l y_l \\ 0 \end{bmatrix},$$

where $\mu_1 y_l + \cdots + \mu_l y_l \in \text{con}\, K$. Hence,

$$\begin{bmatrix} x \\ -\xi \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{G}(\text{con}\, C + \text{con}\, K, \text{con}\, K).$$

[**Case 2:**] $k > 0$ (or $\xi > 0$).
Since we have $\frac{x_i}{\xi_i} \in C$ $(i = 1,\cdots,k)$ and $y_j \in K$ $(j = 1,\cdots,l)$, noting that

$$\xi = \lambda_1\xi_1 + \cdots + \lambda_k\xi_k$$

we have

$$\frac{x}{\xi} = \frac{\lambda_1\xi_1}{\xi}\cdot\frac{x_1}{\xi_1} + \cdots + \frac{\lambda_k\xi_k}{\xi}\cdot\frac{x_k}{\xi_k} + \mu_1 y_1 + \cdots + \mu_l y_l \in \text{con}\, C + \text{con}\, K.$$

Therefore, we have

$$\begin{bmatrix} x \\ -\xi \end{bmatrix} = \xi \begin{bmatrix} \frac{x}{\xi} \\ -1 \end{bmatrix} \in \mathcal{G}(\text{con}\, C + \text{con}\, K, \text{con}\, K),$$

and hence, inclusion $\text{con}\,\mathcal{G}(C, K) \subseteq \mathcal{G}(\text{con}\, C + \text{con}\, K, \text{con}\, K)$ was now shown.

Conversely, suppose that $\begin{bmatrix} x \\ -\xi \end{bmatrix} \in \mathcal{G}(\text{con}\, C + \text{con}\, K, \text{con}\, K)$.

[**Case 1:**] $\xi = 0$.
In this case, since $x \in \text{con}\, K$, we have

$$\exists y_1,\cdots,y_l \in K : x = y_1 + \cdots + y_l.$$

Therefore,

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} y_1 \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} y_l \\ 0 \end{bmatrix} \in \text{con}\,\mathcal{G}(C, K).$$

[**Case 2:**] $\xi > 0$.
Since $\frac{x}{\xi} \in \text{con}\, C + \text{con}\, K$,

$$\exists\lambda_1,\cdots,\lambda_k > 0 \text{ with } \sum_{i=1}^{k} = 1\text{: and } x_i \in C \ (i = 1,\cdots,k),\ y_j \in K \ (j = 1,\cdots,l)$$

such that

$$\frac{x}{\xi} = \lambda_1 x_1 + \cdots + \lambda_k x_k + y_1 + \cdots + y_l.$$

Then, we have

$$\begin{aligned}
\begin{bmatrix} x \\ \xi \end{bmatrix} &= \xi \begin{bmatrix} \frac{x}{\xi} \\ -1 \end{bmatrix} \\
&= \xi\left(\lambda_1 \begin{bmatrix} x_1 \\ -1 \end{bmatrix} + \cdots + \lambda_k \begin{bmatrix} x_k \\ -1 \end{bmatrix} + \begin{bmatrix} y_1 \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} y_l \\ 0 \end{bmatrix}\right) \\
&= \xi\lambda_1 \begin{bmatrix} x_1 \\ -1 \end{bmatrix} + \cdots + \xi\lambda_k \begin{bmatrix} x_k \\ -1 \end{bmatrix} + \xi \begin{bmatrix} y_1 \\ 0 \end{bmatrix} + \cdots + \xi \begin{bmatrix} y_l \\ 0 \end{bmatrix} \\
&\in \text{con}\,\mathcal{G}(C, K).
\end{aligned}$$

□

Exercise (3.42): $C \cup \operatorname{dir} K \subseteq \operatorname{csm} \mathbb{R}^n$ *is convex if and only if*

$$\mathcal{G}(C,K) = \left\{ \lambda \begin{bmatrix} x \\ -1 \end{bmatrix} \mid x \in C, \lambda > 0 \right\} \cup \left\{ \begin{bmatrix} y \\ 0 \end{bmatrix} \mid x \in K \right\}$$

is convex.

(Proof) Suppose $C \cup \operatorname{dir} K$ is convex. Then, by the definitions, C and K are convex and $C + K \subseteq C$. Let $y_1, y_2 \in \mathcal{G}(C,K)$.
[**Case 1:**] $z_i = \lambda_i \begin{bmatrix} x_i \\ -1 \end{bmatrix}$ for some $x_i \in C$ and $\lambda_i > 0$ $(i = 1,2)$.
We have

$$z_1 + z_2 = \lambda_1 \begin{bmatrix} x_1 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} x_2 \\ -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 + \lambda_2 x_2 \\ -\lambda_1 - \lambda_2 \end{bmatrix} = (\lambda_1 + \lambda_2) \begin{bmatrix} \frac{\lambda_1}{\lambda_1+\lambda_2} x_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} x_2 \\ -1 \end{bmatrix} \in \mathcal{G}(C,K)$$

by the convexty of C.
[**Case 2:**] $z_1 = \lambda \begin{bmatrix} x \\ -1 \end{bmatrix}$ for some $x \in C$ and $\lambda > 0$. $z_2 = \begin{bmatrix} y \\ 0 \end{bmatrix}$ for some $y \in K$.
We have

$$z_1 + z_2 = \lambda \begin{bmatrix} x \\ -1 \end{bmatrix} + \begin{bmatrix} y \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x + \frac{y}{\lambda} \\ -1 \end{bmatrix} \in \mathcal{G}(C,K)$$

by $C + K \subseteq C$.
[**Case 3:**] $z_i = \begin{bmatrix} y_i \\ 0 \end{bmatrix}$ for some $y_i \in K$ $(i = 1,2)$.
We have

$$z_1 + z_2 = \begin{bmatrix} y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} y_1 + y_2 \\ 0 \end{bmatrix} \in \mathcal{G}(C,K)$$

by the convexity of K. Therefore, $\mathcal{G}(C,K)$ is convex.

Conversely, suppose that $\mathcal{G}(C,K)$ is convex. Then, it is obvious that C and K are convex. We will show that $C + K \subseteq C$. Let $x \in C$ and $y \in K$. Then, by the convexity of $\mathcal{G}(C,K)$, we have

$$\begin{bmatrix} x + y \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ -1 \end{bmatrix} + \begin{bmatrix} y \\ 0 \end{bmatrix} \in \mathcal{G}(C,K).$$

Hence, $x + y \in C$. □

4.F: Horizon Limit

Exercise (4.20'): *We have*

(i) $\limsup_\nu (C^\nu \cup \operatorname{dir} K^\nu) = (\limsup_\nu C^\nu) \cup \operatorname{dir} (\limsup_\nu^\infty C^\nu \cup \limsup_\nu K^\nu)$,

(ii) $\liminf_\nu (C^\nu \cup \operatorname{dir} K^\nu) \supseteq (\liminf_\nu C^\nu) \cup \operatorname{dir} (\liminf_\nu^\infty C^\nu \cup \liminf_\nu K^\nu)$.

If $K^\nu = \{\mathbf{0}\}$ $(\nu \in N)$, *we have equality in (ii).*

(Proof) (i) Suppose $\bar{x} \in \limsup_\nu (C^\nu \cup \operatorname{dir} K^\nu)$. Then,

$$\exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in C^\nu \cup \operatorname{dir} K^\nu \ (\nu \in N) \colon x^\nu \xrightarrow[N]{} \bar{x}.$$

[**Case I:** $\bar{x} \in \mathbb{R}^n$.] In this case, for a sufficiently large ν_0 we have $\nu \geq \nu_0, \nu \in N$ implies $x^\nu \in C^\nu$. Hence, $\bar{x} \in \limsup_\nu C^\nu$.

[**Case II:** $\bar{x} \in \operatorname{hzn} \mathbb{R}^n$.] Suppose $\bar{x} = \operatorname{dir} x$ for some $x \neq \mathbf{0}$.

[**Case II-a:** There exist infinitely many $\nu \in N$ such that $x^\nu \in C^\nu$.] Then, by definition, there exists $\lambda^\nu \searrow 0$ such that $\lambda^\nu x^\nu \xrightarrow[N]{} x$. Hence, $x \in \limsup_\nu^\infty C^\nu$. Therefore, $\bar{x} = \operatorname{dir} x \in \operatorname{dir} \limsup_\nu^\infty C^\nu$.

[**Case II-b:** There exists infinitely many $\nu \in N$ such that $x^\nu \in \operatorname{dir} K^\nu$.] In this case, for each such ν there exists $y^\nu \in K^\nu$ such that $x^\nu = \operatorname{dir} y^\nu$. Also, we have $\lambda^\nu y^\nu \to x$ for some $\lambda^\nu > 0$ by definition of convergence of direction points. Therefore, $x \in \limsup_\nu K^\nu$, and hence, $\bar{x} \in \operatorname{dir} \limsup_\nu K^\nu$.

We thus have shown that $\limsup_\nu (C^\nu \cup \mathrm{dir}\, K^\nu) \subseteq (\limsup_\nu C^\nu) \cup \mathrm{dir}\,(\limsup_\nu^\infty C^\nu \cup \limsup_\nu K^\nu)$.

Conversely, suppose that $\bar{x} \in (\limsup_\nu C^\nu) \cup \mathrm{dir}\,(\limsup_\nu^\infty C^\nu \cup \limsup_\nu K^\nu)$. If $\bar{x} \in (\limsup_\nu C^\nu)$, we apparently have $\bar{x} \in \limsup_\nu (C^\nu \cup \mathrm{dir}\, K^\nu)$.

If $\bar{x} \in \mathrm{dir}\,(\limsup_\nu^\infty C^\nu \cup \limsup_\nu K^\nu)$, we have $\bar{x} = \mathrm{dir}\, x$ for some $x \in \limsup_\nu^\infty C^\nu \cup \limsup_\nu K^\nu$ with $x \neq \mathbf{0}$. In case of $x \in \limsup_\nu K^\nu$, we have

$$\exists N \in \mathcal{N}_\infty^\#, \exists y^\nu \in K^\nu \ (\nu \in N) \colon y^\nu \underset{N}{\longrightarrow} x,$$

and hence,

$$\exists \mathrm{dir}\, y^\nu \in \mathrm{dir}\, K^\nu \ (\nu \in N) \colon \mathrm{dir}\, y^\nu \underset{N}{\longrightarrow} \mathrm{dir}\, x.$$

Therefore, $\bar{x} = \mathrm{dir}\, x \in \limsup_\nu \mathrm{dir}\, K^\nu$.

In the other case, we have

$$\exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in C^\nu \ (\nu \in N), \exists \lambda^\nu \searrow 0 \colon \lambda^\nu x^\nu \underset{N}{\longrightarrow} x.$$

By definition, we have

$$\exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in C^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} \mathrm{dir}\, x,$$

and hence, we have $\bar{x} = \mathrm{dir}\, x \in \limsup_\nu C^\nu$.

(ii) Suppose $\bar{x} \in (\liminf_\nu C^\nu) \cup \mathrm{dir}\,(\liminf_\nu^\infty C^\nu \cup \liminf_\nu K^\nu)$.

If $\bar{x} \in \liminf_\nu C^\nu$, then it is clear that $x \in \liminf_\nu (C^\nu \cup \mathrm{dir}\, K^\nu)$. Suppose $\bar{x} \in \mathrm{dir}\,(\liminf_\nu^\infty C^\nu \cup \liminf_\nu K^\nu)$. Then, there exists $x \in \liminf_\nu^\infty C^\nu \cup \liminf_\nu K^\nu$ such that $\bar{x} = \mathrm{dir}\, x$.
[Case: $x \in \liminf_\nu^\infty C^\nu$.] We have

$$\exists N \in \mathcal{N}_\infty, \exists x^\nu \in C^\nu \ (\nu \in N), \exists \lambda^\nu \searrow 0 \colon \lambda^\nu x^\nu \underset{N}{\longrightarrow} x,$$

and hence,

$$\exists x^\nu \in C^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} \mathrm{dir}\, x.$$

Therefore, $\bar{x} = \mathrm{dir}\, x \in \liminf_\nu C^\nu$.
[Case: $x \in \liminf_\nu K^\nu$.] We have

$$\exists N \in \mathcal{N}, \exists x^\nu \in K^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} x,$$

that is,

$$\exists \mathrm{dir}\, x^\nu \in \mathrm{dir}\, K^\nu \ (\nu \in N) \colon \mathrm{dir}\, x^\nu \underset{N}{\longrightarrow} \mathrm{dir}\, x.$$

Hence, $\bar{x} = \mathrm{dir}\, x \in \liminf_\nu \mathrm{dir}\, K^\nu$.

(iii) Suppose $K^\nu = \{\mathbf{0}\}$ for each ν and let $\bar{x} \in \liminf_\nu C^\nu$. If $\bar{x} \in \mathbb{R}^n$, then

$$\exists N \in \mathcal{N}_\infty, \exists x^\nu \in C^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} \bar{x},$$

and hence, $\bar{x} \in \liminf_\nu C^\nu$. If $\bar{x} \in \mathrm{hzn}\, \mathbb{R}^n$, then for some $x \neq \mathbf{0}$ we have $\bar{x} = \mathrm{dir}\, x$ and

$$\exists N \in \mathcal{N}_\infty, \exists x^\nu \in C^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} \mathrm{dir}\, x,$$

and hence,

$$\exists x^\nu \in C^\nu \ (\nu \in N), \exists \lambda^\nu \searrow 0 \colon \lambda^\nu x^\nu \underset{N}{\longrightarrow} x.$$

Therefore, $x \in \liminf_\nu^\infty C^\nu$. Then, we have $\bar{x} = \mathrm{dir}\, x \in \mathrm{dir}\, \liminf_\nu^\infty C^\nu$. □

Proposition: *For a general subset of* $\mathrm{csm}\, \mathbb{R}^n$*, written as* $C \cup \mathrm{dir}\, K$ *for a set* $C \subseteq \mathbb{R}^n$ *and a cone* $K \subseteq \mathbb{R}^n$*, let*

$$\mathcal{G}(C, K) = \left\{ \lambda \begin{bmatrix} x \\ -1 \end{bmatrix} \mid x \in C, \lambda > 0 \right\} \cup \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in K \right\}.$$

Then, we have

(i) $\limsup_\nu (C^\nu \cup \mathrm{dir}\, K^\nu) \subseteq C \cup \mathrm{dir}\, K$ *if and only if* $\limsup_\nu \mathcal{G}(C^\nu, K^\nu) \subseteq \mathcal{G}(C, K)$.

(ii) $\liminf_\nu (C^\nu \cup \mathrm{dir}\ K^\nu) \supseteq C \cup \mathrm{dir}\ K$ *if and only if* $\liminf_\nu \mathcal{G}(C^\nu, K^\nu) \supseteq \mathcal{G}(C, K)$.

(iii) $C^\nu \cup \mathrm{dir}\ K^\nu \xrightarrow{\mathrm{c}} C \cup \mathrm{dir}\ K$ *if and only if* $\mathcal{G}(C^\nu, K^\nu) \to \mathcal{G}(C, K)$.

(Proof) (i) ("if" part:) Suppose that $\limsup_\nu \mathcal{G}(C^\nu, K^\nu) \subseteq \mathcal{G}(C, K)$. By Exercise 4.20, it suffices to show that

$$\limsup_\nu C^\nu \subseteq C \text{ and } \limsup_\nu^\infty C_\nu \cup \limsup_\nu K^\nu \subseteq K.$$

The first inclusion follows from the following chain of implications.

$$\begin{aligned}
&\bar{x} \in \limsup_\nu C^\nu && (1)\\
&\Rightarrow \quad \exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in C^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} x && (2)\\
&\Rightarrow \quad \exists N \in \mathcal{N}_\infty^\#, \exists \begin{bmatrix} x^\nu \\ -1 \end{bmatrix} \in \mathcal{G}(C^\nu, K^\nu) \ (\nu \in N) \colon \begin{bmatrix} x^\nu \\ -1 \end{bmatrix} \underset{N}{\longrightarrow} \begin{bmatrix} \bar{x} \\ -1 \end{bmatrix} && (3)\\
&\Rightarrow \quad \begin{bmatrix} \bar{x} \\ -1 \end{bmatrix} \in \limsup_\nu \mathcal{G}(C^\nu, K^\nu) && (4)\\
&\Rightarrow \quad \begin{bmatrix} \bar{x} \\ -1 \end{bmatrix} \in \mathcal{G}(C, K) && (5)\\
&\Rightarrow \quad \bar{x} \in C. && (6)
\end{aligned}$$

The inclusion $\limsup_\nu^\infty C^\nu \subseteq K$ can be shown by the followings.

$$\begin{aligned}
&x \in \limsup_\nu^\infty C^\nu && (7)\\
&\Rightarrow \quad \exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in C^\nu \ (\nu \in N), \exists \lambda^\nu \searrow 0 \colon \lambda^\nu x^\nu \underset{N}{\longrightarrow} \bar{x} && (8)\\
&\Rightarrow \quad \exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in C^\nu \ (\nu \in N), \exists \lambda^\nu > 0 \colon \begin{bmatrix} \lambda^\nu x^\nu \\ -\lambda^\nu \end{bmatrix} \underset{N}{\longrightarrow} \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix} && (9)\\
&\Rightarrow \quad \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix} \in \limsup_\nu \mathcal{G}(C^\nu, K^\nu) && (10)\\
&\Rightarrow \quad \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix} \in \mathcal{G}(C, K) && (11)\\
&\Rightarrow \quad \bar{x} \in K && (12)
\end{aligned}$$

$\limsup_\nu K^\nu \subseteq K$

$$\begin{aligned}
&x \in \limsup_\nu K^\nu && (13)\\
&\Rightarrow \quad \exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in K^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} x && (14)\\
&\Rightarrow \quad \exists N \in \mathcal{N}, \begin{bmatrix} x^\nu \\ 0 \end{bmatrix} \in \mathcal{G}(C^\nu, K^\nu) \ (\nu \in N) \colon \begin{bmatrix} x^\nu \\ 0 \end{bmatrix} \underset{N}{\longrightarrow} \begin{bmatrix} x \\ 0 \end{bmatrix} && (15)\\
&\Rightarrow \quad \begin{bmatrix} x \\ 0 \end{bmatrix} \in \limsup_\nu \mathcal{G}(C^\nu, K^\nu) && (16)\\
&\Rightarrow \quad \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{G}(C, K) && (17)\\
&\Rightarrow \quad x \in K. && (18)
\end{aligned}$$

("only if": part) Suppose that $\limsup_\nu (C^\nu \cup \mathrm{dir}\ K^\nu) \subseteq C \cup \mathrm{dir}\ K$. Let $\tilde{x} = \begin{bmatrix} \bar{x} \\ -\gamma \end{bmatrix} \in \limsup_\nu \mathcal{G}(C^\nu, K^\nu)$. Then, we have

$$\exists N \in \mathcal{N}_\infty^\#, \exists \begin{bmatrix} x^\nu \\ -\gamma^\nu \end{bmatrix} \in \mathcal{G}(C^\nu, K^\nu) \ (\nu \in N) \colon \begin{bmatrix} x^\nu \\ -\gamma^\nu \end{bmatrix} \underset{N}{\longrightarrow} \begin{bmatrix} \bar{x} \\ -\gamma \end{bmatrix}$$

[**Case:** $\gamma > 0$.] We must have for all but finite $\nu \in N$ that $\gamma^\nu > 0$ and $\frac{x^\nu}{\gamma} \in C^\nu$. Therefore, $\frac{\bar{x}}{\gamma} \in \limsup_\nu C^\nu$, and hence, $\frac{\bar{x}}{\gamma} \in C$. Then, $\begin{bmatrix} \bar{x} \\ -\gamma \end{bmatrix} \in \mathcal{G}(C, K)$.

[**Case:** $\gamma = 0$.] Define $N_1 = \{\nu \mid \nu \in N, \gamma^\nu > 0\}$ and $N_0 = \{\nu \mid \nu \in N, \gamma^\nu = 0\}$. If N_1 is infinite, then we have

$$\gamma^\nu \searrow 0, \frac{x^\nu}{\gamma^\nu} \in C^\nu \ (\nu \in N), \gamma^\nu \frac{x^\nu}{\gamma^\nu} \xrightarrow[N_1]{} \bar{x}.$$

If N_0 is infinite, then we have

$$x^\nu \in K^\nu, x^\nu \xrightarrow[N_0]{} \bar{x},$$

and hence, $\operatorname{dir} \bar{x} \in \limsup_\nu \operatorname{dir} K^\nu$. Therefore, we have $\operatorname{dir} \bar{x} \in \limsup_\nu (C^\nu \cup \operatorname{dir} K^\nu) \subseteq \operatorname{dir} K$. Now, we have $\bar{x} \in K$, and hence, $\begin{bmatrix} \bar{x} \\ 0 \end{bmatrix} \in \mathcal{G}(C, K)$.

(ii) ("if" part:) Suppose $\bar{x} \in C \cup \operatorname{dir} K$.

[Case: $\bar{x} \in C$] We have

$$\bar{x} \in C \tag{19}$$
$$\Rightarrow \begin{bmatrix} \bar{x} \\ -1 \end{bmatrix} \in \mathcal{G}(C, K) \tag{20}$$
$$\Rightarrow \begin{bmatrix} \bar{x} \\ -1 \end{bmatrix} \in \liminf_\nu \mathcal{G}(C^\nu, K^\nu) \tag{21}$$
$$\Rightarrow \exists N \in \mathcal{N}_\infty, \exists x^\nu \in C^\nu \ (\nu \in N), \exists \lambda^\nu > 0 \ (\nu \in N)\colon \begin{bmatrix} \lambda^\nu x^\nu \\ -\lambda^\nu \end{bmatrix} \xrightarrow[N]{} \begin{bmatrix} \bar{x} \\ -1 \end{bmatrix} \tag{22}$$
$$\Rightarrow \exists N \in \mathcal{N}_\infty, \exists x^\nu \in C^\nu \ (\nu \in N)\colon x^\nu \xrightarrow[N]{} \bar{x} \tag{23}$$
$$\Rightarrow \bar{x} \in \liminf_\nu C^\nu. \tag{24}$$

[Case $\bar{x} \in \operatorname{dir} K$] Let $x \in K$ be such that $\bar{x} = \operatorname{dir} x$. Then we have

$$\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{G}(C, K) \tag{25}$$
$$\Rightarrow \begin{bmatrix} x \\ 0 \end{bmatrix} \in \liminf_\nu \mathcal{G}(C^\nu, K^\nu) \tag{26}$$
$$\Rightarrow \exists N \in \mathcal{N}_\infty, \exists \begin{bmatrix} x^\nu \\ -\gamma^\nu \end{bmatrix} \in \mathcal{G}(C^\nu, K^\nu) \ (\nu \in N)\colon \begin{bmatrix} x^\nu \\ -\gamma^\nu \end{bmatrix} \xrightarrow[N]{} \begin{bmatrix} x \\ 0 \end{bmatrix}. \tag{27}$$

Define

$$N_1 = \{\nu \mid \nu \in N, \gamma^\nu > 0\}$$

and

$$N_0 = \{\nu \mid \nu \in N, \gamma^\nu = 0\}.$$

We have that

$$\frac{x^\nu}{\gamma^\nu} \in C^\nu$$

for each $\nu \in N_1$ and that

$$\exists x^\nu \in K^\nu$$

for each $\nu \in N_0$.

[**Case:** N_0 is finite.] Then, we have

$$N_1 \in \mathcal{N}, \frac{x^\nu}{\gamma^\nu} \in C^\nu \ (\nu \in N_1), \gamma^\nu \searrow 0\colon \gamma^\nu \frac{x^\nu}{\gamma^\nu} \xrightarrow[N_1]{} \bar{x},$$

i.e., $\bar{x} \in \liminf_\nu^\infty C^\nu$.

[**Case:** N_1 is finite.] Then, we have $N_0 \in \mathcal{N}_\infty$ and $x^\nu \xrightarrow[N_0]{} \bar{x}$, and hence, $\bar{x} \in \liminf_\nu K^\nu$.

[**Case:** both of N_1 and N_0 are infinite.] We have that $\gamma^\nu \frac{x^\nu}{\gamma^\nu} \xrightarrow[N_1]{} \bar{x}$, i.e., $x^\nu \xrightarrow[N_1]{} \operatorname{dir} \bar{x}$ and that $x^\nu \xrightarrow[N_0]{} \bar{x}$, i.e., $\operatorname{dir} x^\nu \xrightarrow[N_0]{} \operatorname{dir} \bar{x}$. Therefore, $\operatorname{dir} \bar{x} \in \liminf_\nu (C^\nu \cup \operatorname{dir} K^\nu)$.

("only if" part:) Suppose $\begin{bmatrix} \bar{x} \\ -\gamma \end{bmatrix} \in \mathcal{G}(C, K)$.

[**Case:** $\gamma > 0$.] Then, we have $\frac{\bar{x}}{\gamma} \in C \subseteq \liminf_\nu C^\nu$.

[**Case:** $\gamma = 0$.] We have $\bar{x} \in K$, and hence, $\operatorname{dir}\bar{x} \in \operatorname{dir} K \subseteq \liminf_\nu (C^\nu \cup \operatorname{dir} K^\nu)$. This means by definition that

$$\exists N \in \mathcal{N}_\infty, \exists \tilde{x}^\nu \in C^\nu \cup \operatorname{dir} K^\nu \ (\nu \in N) \colon \tilde{x}^\nu \underset{N}{\longrightarrow} \operatorname{dir}\bar{x}.$$

Let $N_0 = \{\nu \mid \nu \in N, \tilde{x}^\nu \text{ is a direction point}\}$ and $N_1 = \{\nu \mid \nu \in N, \tilde{x}^\nu \text{ is an ordinary point}\}$. If N_1 is infinite, then we have

$$\tilde{x}^\nu \in C^\nu \ (\nu \in N_1), \ \tilde{x}^\nu \underset{N_1}{\longrightarrow} \operatorname{dir}\bar{x},$$

that is,

$$\exists \lambda^\nu \searrow 0, \lambda^\nu \tilde{x}^\nu \underset{N_1}{\longrightarrow} \bar{x}.$$

Then, we have

$$\begin{bmatrix} \lambda^\nu x^\nu \\ -\lambda^\nu \end{bmatrix} \in \mathcal{G}(C^\nu, K^\nu) \ (\nu \in N_1), \begin{bmatrix} \lambda^\nu x^\nu \\ -\lambda^\nu \end{bmatrix} \underset{N_1}{\longrightarrow} \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix}.$$

If N_0 is infinite, then we have

$$\tilde{x}^\nu \in \operatorname{dir} K^\nu \ (\nu \in N_0), \ \tilde{x} \underset{N_0}{\longrightarrow} \operatorname{dir}\bar{x},$$

that is,

$$\exists x^\nu \in K^\nu \text{ s.t. } \operatorname{dir} x^\nu = \tilde{x}^\nu \ (\nu \in N_0), \exists \lambda^\nu > 0 \ (\nu \in N_0) \colon \lambda^\nu x^\nu \underset{N_0}{\longrightarrow} \bar{x},$$

Then,

$$\begin{bmatrix} \lambda^\nu x^\nu \\ 0 \end{bmatrix} \in \mathcal{G}(C^\nu, K^\nu), \begin{bmatrix} \lambda^\nu x^\nu \\ 0 \end{bmatrix} \underset{N_0}{\longrightarrow} \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix}.$$

Consequently, we have

$$\begin{bmatrix} \bar{x} \\ 0 \end{bmatrix} \in \liminf_\nu \mathcal{G}(C^\nu, K^\nu).$$

(iii) This is clear from (i) and (ii). □

5.C: Local Boundedness

Exercise (5.26): *Let $S \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be osc.*

(a) *$S(C)$ is closed when C is closed and $(S^\infty)^{-1}(\mathbf{0}) \cap C^\infty = \{\mathbf{0}\}$. Then, $S(C)^\infty \subseteq S^\infty(C^\infty)$.*

(b) *$S^{-1}(D)$ is closed when D is closed and $S^\infty(\mathbf{0}) \cap D^\infty = \{\mathbf{0}\}$. Then, $(S^{-1}(D))^\infty \subseteq (S^\infty)^{-1}(D^\infty)$.*

(Proof) It suffices to show (b) only.
(The closedness of $S^{-1}(D)$:)

$((S^{-1}(D))^\infty \subseteq (S^\infty)^{-1}(D^\infty)$:) Suppose that $\bar{x} \in (S^{-1}(D))^\infty$. Then,

$$\exists x^\nu \in S^{-1}(D), \exists \lambda^\nu \searrow 0 \colon \lambda^\nu x^\nu \underset{N}{\longrightarrow} \bar{x},$$

which implies

$$\exists u^\nu \in D, \exists x^\nu \in S^{-1}(u^\nu), \exists \lambda^\nu \searrow 0 \colon \lambda^\nu x^\nu \underset{N}{\longrightarrow} \bar{x}.$$

If $\bar{x} = \mathbf{0}$, then we have $\bar{x} = \mathbf{0} \in (S^\infty)^{-1}(\mathbf{0}) \subseteq (S^\infty)^{-1}(D^\infty)$, and we are done. Hence, we assume $\bar{x} \neq \mathbf{0}$.

[**Case I**: $\{u^\nu\}_N$ is bounded] We have

$$\lambda^\nu u^\nu \to 0 \in D^\infty,$$

and hence, $\bar{x} \in (S^{-1})^\infty(D^\infty)$.

[**Case II**: $\{x^\nu\}_N$ is unbounded] Since $\{(x^\nu, u^\nu)\}_N$ is unbounded, there exists a subsequence $\{(x^\nu, u^\nu)\}_{N'}$ of $\{(x^\nu, u^\nu)\}_N$, $\mu^\nu \searrow 0$ and $(\bar{x}', \bar{u}) \neq \mathbf{0}$ such that $\mu^\nu (x^\nu, u^\nu) \underset{N}{\longrightarrow} (\bar{x}', \bar{u})$. If $\bar{x}' = \mathbf{0}$, then we have $\mathbf{0} \neq \bar{u} \in S^\infty(\mathbf{0}) \cap D^\infty$, a contradiction. Therefore, we have $\bar{x}' \neq \mathbf{0}$. It follows from the lemma in my note RW-5.D that $\bar{x}' = \gamma\bar{x}$ for some $\gamma > 0$. Hence, we can assume, by scaling μ^ν if neccessary, that $\bar{x}' = \bar{x}$. Then, we have $\bar{x} \in (S^{-1})^\infty(\bar{u})$ and $\bar{u} \in D^\infty$, and hence, $\bar{x} \in (S^{-1})^\infty(D^\infty)$. □